

Weighted local Hardy spaces associated to Schrödinger operators

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Abstract. In this paper, we characterize the weighted local Hardy spaces $h_\rho^p(\omega)$ related to the critical radius function ρ and weights $\omega \in A_\infty^{\rho, \infty}(\mathbb{R}^n)$ which locally behave as Muckenhoupt's weights and actually include them, by the local vertical maximal function, the local nontangential maximal function and the atomic decomposition. By the atomic characterization, we also prove the existence of finite atomic decompositions associated with $h_\rho^p(\omega)$. Furthermore, we establish boundedness in $h_\rho^p(\omega)$ of quasi-Banach-valued sublinear operators. As their applications, we establish the equivalence of the weighted local Hardy space $h_\rho^1(\omega)$ and the weighted Hardy space $H_\mathcal{L}^1(\omega)$ associated to Schrödinger operators \mathcal{L} with $\omega \in A_1^{\rho, \infty}(\mathbb{R}^n)$.

1 Introduction

The theory of classical local Hardy spaces, originally introduced by Goldberg [17], plays an important role in various field of analysis and partial differential equations; see [6, 24, 26, 32, 33, 34] and their references. In particular, pseudo-difference operators are bounded on local Hardy spaces $h^p(\mathbb{R}^n)$ for $p \in (0, 1]$, but they are not bounded on Hardy spaces $H^p(\mathbb{R}^n)$ for $p \in (0, 1]$; see [17] (also [33, 34]). In [6], Bui studied the weighted local Hardy space $h_\omega^p(\mathbb{R}^n)$ with $\omega \in A_\infty(\mathbb{R}^n)$, where and in what follows, $A_p(\mathbb{R}^n)$ for $p \in [1, \infty]$ denotes the class of Muckenhoupt's weights; see [8, 15, 18, 26] for their definition and properties.

In [23], Rychkov introduced and studied some properties of the weighted Besov-Lipschitz spaces and Triebel-Lizorkin spaces with weights that are locally in $A_p(\mathbb{R}^n)$ but may grow or decrease exponentially, which contain Hardy spaces. In particular, Rychkov [23] generalized some of theories of weighted local Hardy spaces developed by Bui [6] to $A_\infty^{loc}(\mathbb{R}^n)$ weights, where $A_\infty^{loc}(\mathbb{R}^n)$ weights denote local $A_\infty(\mathbb{R}^n)$ weights which are non-doubling weights, and $A_\infty^{loc}(\mathbb{R}^n)$ weights include $A_\infty(\mathbb{R}^n)$ weights. Recently, Tang [28] established the weighted atomic decomposition characterization of the weighted local Hardy space $h_\omega^p(\mathbb{R}^n)$ with $\omega \in A_\infty^{loc}(\mathbb{R}^n)$ via the

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local grand maximal function, and gave some criterions about boundedness of \mathcal{B}_β -sublinear operators on $h_\omega^p(\mathbb{R}^n)$ which was first introduced in [38]; meanwhile, Tang [28] also proved that pseudo-difference operators are bounded on local Hardy spaces $h_\omega^p(\mathbb{R}^n)$ for $p \in (0, 1]$ by using above criterions and main results in [29]. Furthermore, Yang-Yang [36] extended the main results in [28] to the weighted local Orlicz-Hardy space $h_\omega^\Phi(\mathbb{R}^n)$ case by applying similar methods in [28].

On the other hand, the study of schrödinger operator $L = -\Delta + V$ recently attracted much attention; see [3, 4, 10, 11, 25, 30, 31, 38, 39, 40, 41]. In particular, J. Dziubański and J. Zienkiewicz [10, 11] studied Hardy space $H_\mathcal{L}^1$ associated to Schrödinger operators \mathcal{L} with potential satisfying reverse Hölder inequality. Recently, Bongioanni, etc. [3] introduced new classes of weights, related to Schrödinger operators \mathcal{L} , that is, $A_p^{\rho, \infty}(\mathbb{R}^n)$ weight which are in general larger than Muckenhoupt's (see Section 2 for notions of $A_p^{\rho, \infty}(\mathbb{R}^n)$ weight). Nature, it is a very interesting problem that whether we can give a atomic characterization for weighted Hardy space $H_\mathcal{L}^1(\omega)$ with $\omega \in A_1^{\rho, \infty}(\mathbb{R}^n)$.

The purpose of this paper is to give a positive answer. More precisely, we first introduce the weighted local Hardy spaces $h_\rho^p(\omega)$ with $A_q^{\rho, \infty}(\mathbb{R}^n)$ weights, and establish the atomic characterization of the weighted local Hardy spaces $h_\rho^p(\omega)$ with $\omega \in A_q^{\rho, \infty}(\mathbb{R}^n)$ weights. Then, we establish the equivalence between the weighted local Hardy spaces $h_\rho^1(\omega)$ and the weighted Hardy space $H_\mathcal{L}^1(\omega)$ associated to Schrödinger operator \mathcal{L} with $\omega \in A_1^{\rho, \infty}(\mathbb{R}^n)$. In particular, it should be pointed out that we can not directly obtain the atomic characterization of $H_\mathcal{L}^1(\omega)$ with $A_1^{\rho, \infty}(\mathbb{R}^n)$ weights by using the methods in [10, 11, 12], which forces us to use the above weighted local Hardy spaces $h_\rho^1(\omega)$ theory to overcome the difficulty.

The paper is organized as follows. In Section 2, we review some notions and notations concerning the weight classes $A_p^{\rho, \theta}(\mathbb{R}^n)$ introduced in [3, 30, 31]. In Section 3, we first introduce the weighted local Hardy space $h_{\rho, N}^p(\omega)$ via the local grand maximal function, and then the weighted atomic local Hardy space $h_{\rho, N}^{p, q, s}(\omega)$ for any admissible triplet $(p, q, s)_\omega$ (see Definition 3.4 below), furthermore, we establish the local vertical and the local nontangential maximal function characterizations of $h_{\rho, N}^p(\omega)$ via a local Calderón reproducing formula and some useful estimates established by Rychkov [23]. In Section 4, we establish the Calderón-Zygmund decomposition associated with the grand maximal function. In Section 5, we prove that for any given admissible triplet $(p, q, s)_\omega$, $h_{\rho, N}^p(\omega) = h_{\rho, N}^{p, q, s}(\omega)$ with equivalent norms. It is worth pointing out that we obtain Theorem 5.1 by a way different from the methods in [17, 6], but close to those in [1, 28, 36]. For simplicity, in the rest of this introduction, we denote by $h_\rho^p(\omega)$ the weighted local Hardy space $h_{\rho, N}^p(\omega)$. In Section 6, we prove that $\|\cdot\|_{h_{\rho, \text{fin}}^{p, q, s}(\omega)}$ and $\|\cdot\|_{h_\rho^p(\omega)}$ are equivalent quasi-norms on $h_{\rho, \text{fin}}^{p, q, s}(\omega)$ with $q < \infty$, and we obtain criterions for boundedness of \mathcal{B}_β -sublinear operators in $h_\rho^p(\omega)$. We remark that this extends both the results of Meda-Sjögren-Vallarino [21] and Yang-Zhou [38] to the setting of weighted local Hardy spaces. In

Section 7, we apply the atomic characterization of the weighted local Hardy spaces $h_\rho^1(\omega)$ to establish atomic characterization of weighted Hardy space $H_\mathcal{L}^1(\omega)$ associated to Schrödinger operator \mathcal{L} with $A_1^{\rho,\infty}(\mathbb{R}^n)$ weights.

Throughout this paper, we let C denote constants that are independent of the main parameters involved but whose value may differ from line to line. By $A \sim B$, we mean that there exists a constant $C > 1$ such that $1/C \leq A/B \leq C$. The symbol $A \lesssim B$ means that $A \leq CB$. The symbol $[s]$ for $s \in \mathbb{R}$ denotes the maximal integer not more than s . We also set $\mathbb{N} \equiv \{1, 2, \dots\}$ and $\mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$. The multi-index notation is usual: for $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\partial^\alpha = (\partial/\partial_{x_1})^{\alpha_1} \dots (\partial/\partial_{x_n})^{\alpha_n}$. Given a function g on \mathbb{R}^n , we let $L_g \in \mathbb{Z}_+$ denote the maximal number such that g has vanishing moments up to the order L_g , i.e., $\int x^\alpha g(x) dx = 0$ for all multi-indices α with $|\alpha| \leq L_g$. If no vanishing moments of g , then we put $L_g = -1$.

2 Preliminaries

In this section, we review some notions and notations concerning the weight classes $A_p^{\rho,\theta}(\mathbb{R}^n)$ introduced in [3, 30, 31]. Given $B = B(x, r)$ and $\lambda > 0$, we will write λB for the λ -dilate ball, which is the ball with the same center x and with radius λr . Similarly, $Q(x, r)$ denotes the cube centered at x with side length r (here and below only cubes with sides parallel to the axes are considered), and $\lambda Q(x, r) = Q(x, \lambda r)$. Especially, we will denote $2B$ by B^* , and $2Q$ by Q^* .

Let $\mathcal{L} = -\Delta + V$ be a Schrödinger operator on \mathbb{R}^n , $n \geq 3$, where $V \not\equiv 0$ is a fixed non-negative potential. We assume that V belongs to the reverse Hölder class $RH_s(\mathbb{R}^n)$ for some $s \geq n/2$; that is, there exists $C = C(s, V) > 0$ such that

$$\left(\frac{1}{|B|} \int_B V(x)^s dx \right)^{\frac{1}{s}} \leq C \left(\frac{1}{|B|} \int_B V(x) dx \right),$$

for every ball $B \subset \mathbb{R}^n$. Trivially, $RH_q(\mathbb{R}^n) \subset RH_p(\mathbb{R}^n)$ provided $1 < p \leq q < \infty$. It is well known that, if $V \in RH_q(\mathbb{R}^n)$ for some $q > 1$, then there exists $\varepsilon > 0$, which depends only on d and the constant C in above inequality, such that $V \in RH_{q+\varepsilon}(\mathbb{R}^n)$ (see [16]). Moreover, the measure $V(x) dx$ satisfies the doubling condition:

$$\int_{B(y, 2r)} V(x) dx \leq C \int_{B(y, r)} V(x) dx.$$

With regard to the Schrödinger operator \mathcal{L} , we know that the operators derived from \mathcal{L} behave "locally" quite similar to those corresponding to the Laplacian (see [9, 25]). The notion of locality is given by the critical radius function

$$\rho(x) = \frac{1}{m_V(x)} = \sup_{r>0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x, r)} V(y) dy \leq 1 \right\}. \quad (2.1)$$

Throughout the paper we assume that $V \not\equiv 0$, so that $0 < \rho(x) < \infty$ (see [25]). In particular, $m_V(x) = 1$ with $V = 1$ and $m_V(x) \sim (1 + |x|)$ with $V = |x|^2$.

Lemma 2.1. (see [25]) *There exist $C_0 \geq 1$ and $k_0 \geq 1$ so that for all $x, y \in \mathbb{R}^n$*

$$C_0^{-1} \rho(x) \left(1 + \frac{|x - y|}{\rho(x)}\right)^{-k_0} \leq \rho(y) \leq C_0 \rho(x) \left(1 + \frac{|x - y|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}}. \quad (2.2)$$

In particular, $\rho(x) \sim \rho(y)$ when $y \in B(x, r)$ and $r \leq C\rho(x)$, where C is a positive constant.

A ball of the form $B(x, \rho(x))$ is called critical, and in what follows we will call critical radius function to any positive continuous function ρ that satisfies (2.1), not necessarily coming from a potential V . Clearly, if ρ is such a function, so it is $\beta\rho$ for any $\beta > 0$. As the consequence of the above lemma we acquire the following result:

Lemma 2.2. (see [10]) *There exists a sequence of points $x_j \in \mathbb{R}^n$, $j \geq 1$, such that the family $B_j = B(x_j, \rho(x_j))$, $j \geq 1$ satisfies:*

- (a) $\bigcup_j B_j = \mathbb{R}^n$.
- (b) *For every $\sigma \geq 1$ there exist constants C and N_1 such that $\sum_j \chi_{\sigma B_j} \leq C\sigma^{N_1}$.*

In this paper, we write $\Psi_\theta(B) = (1 + r/\rho(x_0))^\theta$, where $\theta \geq 0$, x_0 and r denotes the center and radius of B respectively.

A weight always refers to a positive function which is locally integrable. As in [3], we say that a weight ω belongs to the class $A_p^{\rho, \theta}(\mathbb{R}^n)$ for $1 < p < \infty$, if there is a constant C such that for all balls B

$$\left(\frac{1}{\Psi_\theta(B)|B|} \int_B \omega(y) dy \right) \left(\frac{1}{\Psi_\theta(B)|B|} \int_B \omega^{-\frac{1}{p-1}}(y) dy \right)^{p-1} \leq C.$$

We also say that a nonnegative function ω satisfies the $A_1^{\rho, \theta}(\mathbb{R}^n)$ condition if there exists a constant C such that

$$M_{V, \theta}(\omega)(x) \leq C\omega(x), \text{ a.e. } x \in \mathbb{R}^n.$$

where

$$M_{V, \theta} f(x) \equiv \sup_{x \in B} \frac{1}{\Psi_\theta(B)|B|} \int_B |f(y)| dy.$$

When $V = 0$, we denote $M_0 f(x)$ by $Mf(x)$ (the standard Hardy-Littlewood maximal function). It is easy to see that $|f(x)| \leq M_{V, \theta} f(x) \leq Mf(x)$ for a.e. $x \in \mathbb{R}^n$ and any $\theta \geq 0$.

Clearly, the classes $A_p^{\rho, \theta}$ are increasing with θ , and we denote $A_p^{\rho, \infty} = \bigcup_{\theta \geq 0} A_p^{\rho, \theta}$. By Hölder's inequality, we see that $A_{p_1}^{\rho, \theta} \subset A_{p_2}^{\rho, \theta}$, if $1 \leq p_1 < p_2 < \infty$, and we also denote

$A_\infty^{\rho,\infty} = \bigcup_{p \geq 1} A_p^{\rho,\infty}$. In addition, for $1 \leq p \leq \infty$, denote by p' the adjoint number of p , i.e. $1/p + 1/p' = 1$.

Since $\Psi_\theta(B) \geq 1$ with $\theta \geq 0$, then $A_p \subset A_p^{\rho,\theta}$ for $1 \leq p < \infty$, where A_p denotes the classical Muckenhoupt weights; see [15] and [22]. Moreover, the inclusions are proper. In fact, as the example given in [30], let $\theta > 0$ and $0 \leq \gamma \leq \theta$, it is easy to check that $\omega(x) = (1 + |x|)^{-(n+\gamma)} \notin A_\infty = \bigcup_{p \geq 1} A_p$ and $\omega(x)dx$ is not a doubling measure, but $\omega(x) = (1 + |x|)^{-(n+\gamma)} \in A_1^{\rho,\theta}$ provided that $V = 1$ and $\Psi_\theta(B(x_0, r)) = (1 + r)^\theta$.

In what follows, given a Lebesgue measurable set E and a weight ω , $|E|$ will denote the Lebesgue measure of E and $\omega(E) := \int_E \omega(x) dx$. For any $\omega \in A_\infty^{\rho,\infty}$, the space $L_\omega^p(\mathbb{R}^n)$ with $p \in (0, \infty)$ denotes the set of all measurable functions f such that

$$\|f\|_{L_\omega^p(\mathbb{R}^n)} \equiv \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx \right)^{1/p} < \infty,$$

and $L_\omega^\infty(\mathbb{R}^n) \equiv L^\infty(\mathbb{R}^n)$. The symbol $L_\omega^{1,\infty}(\mathbb{R}^n)$ denotes the set of all measurable functions f such that

$$\|f\|_{L_\omega^{1,\infty}(\mathbb{R}^n)} \equiv \sup_{\lambda > 0} \{ \lambda \omega(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}) \} < \infty.$$

We define the local Hardy-Littlewood maximal operator by

$$M^{loc} f(x) \equiv \sup_{\substack{x \in B(x_0, r) \\ r \leq \rho(x_0)}} \frac{1}{|B|} \int_B |f(y)| dy. \quad (2.3)$$

We remark that balls can be replaced by cubes in definition of $A_p^{\rho,\theta}$ and $M_{V,\theta}$, since $\Psi(B) \leq \Psi(2B) \leq 2^\theta \Psi(B)$. In fact, for the cube $Q = Q(x_0, r)$, we can also define $\Psi_\theta(Q) = (1 + r/\rho(x_0))^\theta$. Then we give the weighted boundedness of $M_{V,\theta}$.

Lemma 2.3. (see [30]) *Let $1 < p < \infty$, $p' = p/(p-1)$ and assume that $\omega \in A_p^{\rho,\theta}$. There exists a constant $C > 0$ such that*

$$\|M_{V,p'\theta} f\|_{L_\omega^p(\mathbb{R}^n)} \leq C \|f\|_{L_\omega^p(\mathbb{R}^n)}.$$

Next, we give some properties of weights class $A_p^{\rho,\theta}$ for $p \geq 1$.

Lemma 2.4. *Let $\omega \in A_p^{\rho,\infty} = \bigcup_{\theta \geq 0} A_p^{\rho,\theta}$ for $p \geq 1$. Then*

- (i) *If $1 \leq p_1 < p_2 < \infty$, then $A_{p_1}^{\rho,\theta} \subset A_{p_2}^{\rho,\theta}$.*
- (ii) *$\omega \in A_p^{\rho,\theta}$ if and only if $\omega^{-\frac{1}{p-1}} \in A_{p'}^{\rho,\theta}$, where $1/p + 1/p' = 1$.*
- (iii) *If $\omega \in A_p^{\rho,\infty}$, $1 < p < \infty$, then there exists $\epsilon > 0$ such that $\omega \in A_{p-\epsilon}^{\rho,\infty}$.*

(iv) Let $f \in L_{loc}(\rho)$, $0 < \delta < 1$, then $(M_{V,\theta}f)^\delta \in A_1^{\rho,\theta}$.

(v) Let $1 < p < \infty$, then $\omega \in A_p^{\rho,\infty}$ if and only if $\omega = \omega_1 \omega_2^{1-p}$, where $\omega_1, \omega_2 \in A_1^{\rho,\infty}$.

(vi) For $\omega \in A_p^{\rho,\theta}$, $Q = Q(x, r)$ and $\lambda > 1$, there exists a positive constant C such that

$$\omega(\lambda Q) \leq C(\Psi_\theta(\lambda Q))^p \lambda^{np} \omega(Q).$$

(vii) If $p \in (1, \infty)$ and $\omega \in A_p^{\rho,\theta}(\mathbb{R}^n)$, then the local Hardy-Littlewood maximal operator M^{loc} is bounded on $L_\omega^p(\mathbb{R}^n)$.

(viii) If $\omega \in A_1^{\rho,\theta}(\mathbb{R}^n)$, then M^{loc} is bounded from $L_\omega^1(\mathbb{R}^n)$ to $L_\omega^{1,\infty}(\mathbb{R}^n)$.

Proof. (i)-(viii) have been proved in [3, 31]. □

For any $\omega \in A_\infty^{\rho,\infty}(\mathbb{R}^n)$, define the critical index of ω by

$$q_\omega \equiv \inf \{p \in [1, \infty) : \omega \in A_p^{\rho,\infty}(\mathbb{R}^n)\}. \quad (2.4)$$

Obviously, $q_\omega \in [1, \infty)$. If $q_\omega \in (1, \infty)$, then $\omega \notin A_{q_\omega}^{\rho,\infty}$.

The symbols $\mathcal{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$, $\mathcal{D}'(\mathbb{R}^n)$ is the dual space of $\mathcal{D}(\mathbb{R}^n)$, and for $\mathcal{D}(\mathbb{R}^n)$, $\mathcal{D}'(\mathbb{R}^n)$ and $L_\omega^p(\mathbb{R}^n)$, we have the following conclusions.

Lemma 2.5. Let $\omega \in A_\infty^{\rho,\infty}(\mathbb{R}^n)$, q_ω be as in (2.4) and $p \in (q_\omega, \infty]$.

- (i) If $\frac{1}{p} + \frac{1}{p'} = 1$, then $\mathcal{D}(\mathbb{R}^n) \subset L_{\omega^{-1/(p-1)}}^{p'}(\mathbb{R}^n)$.
- (ii) $L_\omega^p(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$ and the inclusion is continuous.

By the same method as the proof of Lemma 2.2 in [28], we can get the Lemma 2.5, and we omit the details here.

For any $\varphi \in \mathcal{D}(\mathbb{R}^n)$, let $\varphi_t(x) = t^{-n}\varphi(x/t)$ for $t > 0$ and $\varphi_j(x) = 2^{jn}\varphi(2^jx)$ for $j \in \mathbb{Z}$. It is easy to see that we have the following results.

Lemma 2.6. (see [28]) Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \varphi(x)dx = 1$.

- (i) For any $\Phi \in \mathcal{D}(\mathbb{R}^n)$ and $f \in \mathcal{D}'(\mathbb{R}^n)$, $\Phi * \varphi_t \rightarrow \Phi$ in $\mathcal{D}(\mathbb{R}^n)$ as $t \rightarrow 0$, and $f * \varphi_t \rightarrow f$ in $\mathcal{D}'(\mathbb{R}^n)$ as $t \rightarrow 0$.
- (ii) Let $\omega \in A_\infty^{\rho,\infty}$ and q_ω be as in (2.3). If $q \in (q_\omega, \infty)$, then for any $f \in L_\omega^q(\mathbb{R}^n)$, $f * \varphi_t \rightarrow f$ in $L_\omega^q(\mathbb{R}^n)$ as $t \rightarrow 0$.

3 Weighted local Hardy spaces and their maximal function characterizations

In this section, we introduce the weighted local Hardy spaces $h_{\rho,N}^p(\omega)$ via the local grand maximal function and establish its local vertical and nontangential maximal function characterizations via a local Calderón reproducing formula. We also introduce the weighted atomic local Hardy space $h_{\rho}^{p,q,s}(\omega)$ and give some basic properties of these spaces.

We first introduce some local maximal functions. For $N \in \mathbb{Z}_+$ and $R \in (0, \infty)$, let

$$\mathcal{D}_{N,R}(\mathbb{R}^n) \equiv \left\{ \varphi \in \mathcal{D}(\mathbb{R}^n) : \text{supp}(\varphi) \subset B(0, R), \right. \\ \left. \|\varphi\|_{\mathcal{D}_{N,R}(\mathbb{R}^n)} \equiv \sup_{x \in \mathbb{R}^n} \sup_{\alpha \in \mathbb{Z}_+^n, |\alpha| \leq N} |\partial^\alpha \varphi(x)| \leq 1 \right\}.$$

Definition 3.1. Let $N \in \mathbb{Z}_+$ and $R \in (0, \infty)$. For any $f \in \mathcal{D}'(\mathbb{R}^n)$, the local nontangential grand maximal function $\widetilde{\mathcal{M}}_{N,R}(f)$ of f is defined by setting, for all $x \in \mathbb{R}^n$,

$$\widetilde{\mathcal{M}}_{N,R}(f)(x) \equiv \sup \left\{ |\varphi_l * f(z)| : |x - z| < 2^{-l} < \rho(x), \varphi \in \mathcal{D}_{N,R}(\mathbb{R}^n) \right\}, \quad (3.1)$$

and the local vertical grand maximal function $\mathcal{M}_{N,R}(f)$ of f is defined by setting, for all $x \in \mathbb{R}^n$,

$$\mathcal{M}_{N,R}(f)(x) \equiv \sup \left\{ |\varphi_l * f(x)| : 0 < 2^{-l} < \rho(x), \varphi \in \mathcal{D}_{N,R}(\mathbb{R}^n) \right\}. \quad (3.2)$$

For convenience's sake, when $R = 1$, we denote $\mathcal{D}_{N,R}(\mathbb{R}^n)$, $\widetilde{\mathcal{M}}_{N,R}(f)$ and $\mathcal{M}_{N,R}(f)$ simply by $\mathcal{D}_N^0(\mathbb{R}^n)$, $\widetilde{\mathcal{M}}_N^0(f)$ and $\mathcal{M}_N^0(f)$, respectively; when $R = \max\{R_1, R_2, R_3\} > 1$ (in which R_1, R_2 and R_3 are defined as in Lemma 4.2, 4.4 and 4.8), we denote $\mathcal{D}_{N,R}(\mathbb{R}^n)$, $\widetilde{\mathcal{M}}_{N,R}(f)$ and $\mathcal{M}_{N,R}(f)$ simply by $\mathcal{D}_N(\mathbb{R}^n)$, $\widetilde{\mathcal{M}}_N(f)$ and $\mathcal{M}_N(f)$, respectively. For any $N \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$, obviously,

$$\mathcal{M}_N^0(f)(x) \leq \mathcal{M}_N(f)(x) \leq \widetilde{\mathcal{M}}_N(f)(x).$$

For the local grand maximal function $\mathcal{M}_N^0(f)$, we have the following Proposition 3.1, which can be proved by the same method as in [28, Proposition 2.2]. Here and in what follows, the space $L_{\text{loc}}^1(\mathbb{R}^n)$ denotes the set of all locally integrable functions on \mathbb{R}^n .

Proposition 3.1. Let $N \geq 2$. Then

- (i) There exists a positive constant C such that for all $f \in L_{\text{loc}}^1(\mathbb{R}^n) \cap \mathcal{D}'(\mathbb{R}^n)$ and almost every $x \in \mathbb{R}^n$,

$$|f(x)| \leq \mathcal{M}_N^0(f)(x) \leq CM^{\text{loc}}(f)(x).$$

- (ii) If $\omega \in A_p^{\rho, \theta}(\mathbb{R}^n)$ with $p \in (1, \infty)$, then $f \in L_\omega^p(\mathbb{R}^n)$ if and only if $f \in \mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{M}_N^0(f) \in L_\omega^p(\mathbb{R}^n)$; moreover,

$$\|f\|_{L_\omega^p(\mathbb{R}^n)} \sim \|\mathcal{M}_N^0(f)\|_{L_\omega^p(\mathbb{R}^n)}.$$

- (iii) If $\omega \in A_1^{\rho, \theta}(\mathbb{R}^n)$, then \mathcal{M}_N^0 is bounded from $L_\omega^1(\mathbb{R}^n)$ to $L_\omega^{1, \infty}(\mathbb{R}^n)$.

Now we introduce the weighted local Hardy space via the local grand maximal function as follows.

Definition 3.2. Let $\omega \in A_\infty^{\rho, \infty}(\mathbb{R}^n)$, q_ω be as in (2.4), $p \in (0, 1]$ and $\tilde{N}_{p, \omega} \equiv [n(\frac{q_\omega}{p} - 1)] + 2$. For each $N \in \mathbb{N}$ with $N \geq \tilde{N}_{p, \omega}$, the weighted local Hardy space is defined by

$$h_{\rho, N}^p(\omega) \equiv \{f \in \mathcal{D}'(\mathbb{R}^n) : \mathcal{M}_N(f) \in L_\omega^p(\mathbb{R}^n)\}.$$

Moreover, let $\|f\|_{h_{\rho, N}^p(\omega)} \equiv \|\mathcal{M}_N(f)\|_{L_\omega^p(\mathbb{R}^n)}$.

Obviously, for any integers N_1 and N_2 with $N_1 \geq N_2 \geq \tilde{N}_{p, \omega}$,

$$h_{\rho, \tilde{N}_{p, \omega}}^p(\omega) \subset h_{\rho, N_2}^p(\omega) \subset h_{\rho, N_1}^p(\omega),$$

and the inclusions are continuous.

Next, we introduce the weighted local atoms, via which, we give the definition of the weighted atomic local Hardy space.

Definition 3.3. Let $\omega \in A_\infty^{\rho, \infty}(\mathbb{R}^n)$, q_ω be as in (2.4). A triplet $(p, q, s)_\omega$ is called to be admissible, if $p \in (0, 1]$, $q \in (q_\omega, \infty]$ and $s \in \mathbb{N}$ with $s \geq [n(q_\omega/p - 1)]$. A function a on \mathbb{R}^n is said to be a $(p, q, s)_\omega$ -atom if

- (i) $\text{supp } a \subset Q(x, r)$ and $r \leq L_1 \rho(x)$,
- (ii) $\|a\|_{L_\omega^q(\mathbb{R}^n)} \leq [\omega(Q)]^{1/q-1/p}$,
- (iii) $\int_{\mathbb{R}^n} a(x) x^\alpha dx = 0$ for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq s$, when $Q = Q(x, r)$, $r < L_2 \rho(x)$,

where $L_1 \equiv 4C_0(3\sqrt{n})^{k_0}$, $L_2 \equiv 1/C_0^2(3\sqrt{n})^{k_0+1}$, and C_0, k_0 are constant given in Lemma 2.1. Moreover, a function $a(x)$ on \mathbb{R}^n is called a $(p, q)_\omega$ -single-atom with $q \in (q_\omega, \infty]$, if

$$\|a\|_{L_\omega^q(\mathbb{R}^n)} \leq [\omega(\mathbb{R}^n)]^{1/q-1/p}.$$

Definition 3.4. Let $\omega \in A_\infty^{\rho, \infty}(\mathbb{R}^n)$, q_ω be as in (2.4), and $(p, q, s)_\omega$ be admissible, The weighted atomic local Hardy space $h_p^{p, q, s}(\omega)$ is defined as the set of all $f \in \mathcal{D}'(\mathbb{R}^n)$ satisfying that

$$f = \sum_{i=0}^{\infty} \lambda_i a_i$$

in $\mathcal{D}'(\mathbb{R}^n)$, where $\{a_i\}_{i \in \mathbb{N}}$ are $(p, q, s)_\omega$ -atoms with $\text{supp}(a_i) \subset Q_i$, a_0 is a $(p, q)_\omega$ -single-atom, $\{\lambda_i\}_{i \in \mathbb{Z}_+} \subset \mathbb{C}$. Moreover, the quasi-norm of $f \in h_\rho^{p,q,s}(\omega)$ is defined by

$$\|f\|_{h_\rho^{p,q,s}(\omega)} \equiv \inf \left\{ \left[\sum_{i=0}^{\infty} |\lambda_i|^p \right]^{1/p} \right\},$$

where the infimum is taken over all the decompositions of f as above.

It is easy to see that if triplets $(p, q, s)_\omega$ and $(p, \bar{q}, \bar{s})_\omega$ are admissible and satisfy $\bar{q} \leq q$ and $\bar{s} \leq s$, then $(p, q, s)_\omega$ -atoms are $(p, \bar{q}, \bar{s})_\omega$ -atoms, which implies that $h_\rho^{p,q,s}(\omega) \subset h_\rho^{p,\bar{q},\bar{s}}(\omega)$ and the inclusion is continuous.

Next, we introduce some local vertical, tangential and nontangential maximal functions, and then we establish the characterizations of the weighted local Hardy space $h_{\rho,N}^p(\omega)$ by these local maximal functions.

Definition 3.5. *Let*

$$\psi_0 \in \mathcal{D}(\mathbb{R}^n) \text{ with } \int_{\mathbb{R}^n} \psi_0(x) dx \neq 0. \quad (3.3)$$

For every $x \in \mathbb{R}^n$, there exists an integer $j_x \in \mathbb{Z}$ satisfying $2^{-j_x} < \rho(x) \leq 2^{-j_x+1}$, and then for $j \geq j_x$, $A, B \in [0, \infty)$ and $y \in \mathbb{R}^n$, let $m_{j,A,B,x}(y) \equiv (1 + 2^j|y|)^A 2^{B|y|/\rho(x)}$.

The local vertical maximal function $\psi_0^+(f)$ of f associated to ψ_0 is defined by setting, for all $x \in \mathbb{R}^n$,

$$\psi_0^+(f)(x) \equiv \sup_{j \geq j_x} |(\psi_0)_j * f(x)|, \quad (3.4)$$

the local tangential Peetre-type maximal function $\psi_{0,A,B}^{**}(f)$ of f associated to ψ_0 is defined by setting, for all $x \in \mathbb{R}^n$,

$$\psi_{0,A,B}^{**}(f)(x) \equiv \sup_{j \geq j_x, y \in \mathbb{R}^n} \frac{|(\psi_0)_j * f(x-y)|}{m_{j,A,B,x}(y)}, \quad (3.5)$$

and the local nontangential maximal function $(\psi_0)_\nabla^*(f)$ of f associated to ψ_0 is defined by setting, for all $x \in \mathbb{R}^n$,

$$(\psi_0)_\nabla^*(f)(x) \equiv \sup_{|x-y| < 2^{-l} < \rho(x)} |(\psi_0)_l * f(y)|, \quad (3.6)$$

where $l \in \mathbb{Z}$.

Obviously, for any $x \in \mathbb{R}^n$, we have

$$\psi_0^+(f)(x) \leq (\psi_0)_\nabla^*(f)(x) \lesssim \psi_{0,A,B}^{**}(f)(x).$$

We point out that the local tangential Peetre-type maximal function $\psi_{0,A,B}^{**}(f)$ was introduced by Rychkov [23].

In order to characterize $h_{\rho,N}^p(\omega)$ by the local vertical and the local nontangential maximal function, we need to establish some relations in the norm of $L_\omega^p(\mathbb{R}^n)$ of the local maximal functions $\psi_{0,A,B}^{**}(f)$, $\psi_0^+(f)$ and $\widetilde{\mathcal{M}}_{N,R}(f)$, which further imply the desired characterizations.

We begin with a lemma on local reproducing formula, which can be deduced from the Lemma 1.6 in [23], and we omit the details of its proof here.

Lemma 3.1. *Let ψ_0 be as in (3.3) and $\psi(x) \equiv \psi_0(x) - (1/2^n)\psi_0(x/2)$ for all $x \in \mathbb{R}^n$. Then for any given integers $j \in \mathbb{Z}$ and $L \in \mathbb{Z}_+$, there exist $\varphi_0, \varphi \in \mathcal{D}(\mathbb{R}^n)$ such that $L_\varphi \geq L$ and*

$$f = (\varphi_0)_j * (\psi_0)_j * f + \sum_{k=j+1}^{\infty} \varphi_k * \psi_k * f \quad (3.7)$$

in $\mathcal{D}'(\mathbb{R}^n)$ for all $f \in \mathcal{D}'(\mathbb{R}^n)$.

Lemma 3.2. *Let $0 < r < \infty$, ψ_0 be as in (3.3) and $\psi(x) \equiv \psi_0(x) - (1/2^n)\psi_0(x/2)$. Then there exists a positive constant A_0 depending only on the support of ψ_0 such that for any $A \in (A_0, \infty)$ and $B \in [0, \infty)$, there exists a positive constant C depending only on n, r, ψ_0, A and B , such that for all $f \in \mathcal{D}'(\mathbb{R}^n)$, $x, x_0 \in \mathbb{R}^n$ and $j \geq j_{x_0}$ (where $2^{-j_{x_0}} < \rho(x_0) \leq 2^{-j_{x_0}+1}$), we have*

$$|\psi_j * f(x)|^r \leq C \sum_{k=j}^{\infty} 2^{(j-k)Ar} 2^{kn} \int \frac{|\psi_k * f(x-y)|^r}{m_{j,A,B,r,x_0}(y)} dy. \quad (3.8)$$

Proof. By Lemma 3.1, we can find $\varphi_0, \varphi \in \mathcal{D}(\mathbb{R}^n)$ so that $L_\varphi \geq A$ and (3.7) is true. Hence, we have

$$\psi_j * f = (\varphi_0)_j * (\psi_0)_j * \psi_j * f + \sum_{k=j+1}^{\infty} \psi_j * \varphi_k * \psi_k * f. \quad (3.9)$$

The function $\psi_j * \varphi_k$ ($k \geq j+1$) have support size $\leq C2^{-j}$ and enjoy the uniform estimate

$$\|\psi_j * \varphi_k\|_{L^\infty(\mathbb{R}^n)} \leq C2^{(j-k)A} 2^{jn}, \quad (3.10)$$

which can be easily deduced by the moment condition on φ (see [23, (2.13)]). Therefore, we may write

$$|\psi_j * \varphi_k(y)| \leq C \frac{2^{(j-k)A} 2^{kn}}{m_{j,A,B,x_0}(y)} \quad (y \in \mathbb{R}^n). \quad (3.11)$$

Putting (3.11) together with the similar estimate for $(\varphi_0)_j * (\psi_0)_j$ into (3.9) gives (3.8) for $r = 1$, and the case $r > 1$ follows by Hölder's inequality. To obtain the case $r < 1$, we introduce the maximal functions

$$M_{A,B,x_0}(x, j) = \sup_{k \geq j, y \in \mathbb{R}^n} 2^{(j-k)A} \frac{|\psi_k * f(x-y)|}{m_{j,A,B,x_0}(y)}.$$

The (3.8) with $r = 1$ gives

$$2^{(j-k)A} |\psi_k * f(x-y)| \leq C \sum_{l=k}^{\infty} 2^{(j-l)A} 2^{ln} \int \frac{|\psi_l * f(x-z)|}{m_{k,A,B,x_0}(z-y)} dz, \quad (3.12)$$

and the right of (3.12) decreases as k increases. Hence, to get the estimate for $M_{A,B,x_0}(x,j)$, we may only consider (3.12) with $k = j$. Combing with the elementary inequality

$$m_{j,A,B,x_0}(z) \leq m_{j,A,B,x_0}(y)m_{k,A,B,x_0}(z-y). \quad (3.13)$$

we can get

$$\begin{aligned} M_{A,B,x_0}(x,j) &\leq C \sum_{k=j}^{\infty} 2^{(j-k)A} 2^{kn} \int \frac{|\psi_l * f(x-z)|}{m_{j,A,B,x_0}(z)} dz \\ &\leq CM_{A,B,x_0}(x,j)^{1-r} \sum_{k=j}^{\infty} 2^{(j-k)Ar} 2^{kn} \int \frac{|\psi_l * f(x-z)|^r}{m_{j,A,B,x_0}(z)^r} dz. \end{aligned} \quad (3.14)$$

Considering $|\psi_j * f(x)| \leq M_{A,B,x_0}(x,j)$, (3.14) implies (3.8), if $M_{A,B,x_0}(x,j) < \infty$. By [18, Proposition 2.3.4(a)], for any $f \in \mathcal{D}'(\mathbb{R}^n)$, we have $M_{A,B,x_0}(x,j) < \infty$ for all $x \in \mathbb{R}^n$ and $j \geq j_{x_0}$, provided $A > A_0$, where A_0 is a positive constant depending only on the support of ψ_0 . This finishes the proof. \square

For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $B \in [0, \infty)$ and $x \in \mathbb{R}^n$, let

$$K_B f(x) = \frac{1}{(\rho(x))^n} \int_{\mathbb{R}^n} |f(y)| 2^{-B \frac{|x-y|}{\rho(x)}} dy, \quad (3.15)$$

and for the operator K_B , we have the following lemma:

Lemma 3.3. *Let $p \in (1, \infty)$ and $\omega \in A_p^{\rho, \theta}(\mathbb{R}^n)$, then there exist constants $C > 0$ and $B_0 \equiv B_0(\omega, n) > 0$ such that for all $B > B_0/p$,*

$$\|K_B f\|_{L^p_{\omega}(\mathbb{R}^n)} \leq C \|f\|_{L^p_{\omega}(\mathbb{R}^n)},$$

for all $f \in L^p_{\omega}(\mathbb{R}^n)$.

Proof. It is suffice to show that there exists a constant $C > 0$ such that for all $B > B_0$,

$$K_B f(x) \leq CM_{V,p'\theta} f(x),$$

then combining with Lemma 2.3, we get the boundedness of the operator K_B .

To control $K_B f(x)$, we argue as follows:

$$\begin{aligned} K_B f(x) &= \frac{1}{(\rho(x))^n} \int_{\mathbb{R}^n} |f(y)| 2^{-B \frac{|x-y|}{\rho(x)}} dy \\ &= \frac{1}{(\rho(x))^n} \int_{|y-x| < \rho(x)} |f(y)| 2^{-B \frac{|x-y|}{\rho(x)}} dy + \frac{1}{(\rho(x))^n} \int_{|y-x| \geq \rho(x)} |f(y)| 2^{-B \frac{|x-y|}{\rho(x)}} dy \\ &= \frac{1}{(\rho(x))^n} \int_{|y-x| < \rho(x)} |f(y)| 2^{-B \frac{|x-y|}{\rho(x)}} dy \\ &\quad + \sum_{k=0}^{\infty} \frac{1}{(\rho(x))^n} \int_{|y-x| \sim 2^k \rho(x)} |f(y)| 2^{-B \frac{|x-y|}{\rho(x)}} dy \\ &\equiv I_1 + I_2. \end{aligned}$$

For I_1 , it is easy to get

$$I_1 \leq \frac{C}{\Psi_{p'\theta}(B_1)|B_1|} \int_{B_1} |f(y)| dy \leq CM_{V,p'\theta}f(x),$$

in which $B_1 = B(x, \rho(x))$ is a critical ball.

For I_2 , we have

$$\begin{aligned} I_2 &\leq C \sum_{k=0}^{\infty} \frac{(1 + 2^{k+1})^{p'\theta} 2^{kn}}{2^{B2^k}} \frac{1}{\Psi_{p'\theta}(2^{k+1}B_1)|2^{k+1}B_1|} \int_{2^{k+1}B_1} |f(y)| dy \\ &\leq C \left(\sum_{k=0}^{\infty} \frac{(1 + 2^{k+1})^{p'\theta} 2^{kn}}{2^{B2^k}} \right) M_{V,p'\theta}f(x) \\ &\leq CM_{V,p'\theta}f(x), \end{aligned}$$

where the sum converges when $B > B_0/p$. \square

Lemma 3.4. *Let ψ_0 be as in (3.3) and $r \in (0, \infty)$. Then for any $A \in (\max\{A_0, n/r\}, \infty)$ (where A_0 is as in Lemma 3.2) and $B \in [0, \infty)$, there exists a positive constant C , depending only on n, r, ψ_0, A and B , such that for all $f \in \mathcal{D}'(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $j \geq j_x$ (where $2^{-j_x} < \rho(x) \leq 2^{-j_x+1}$),*

$$\begin{aligned} [(\psi_0)_{j,A,B}^*(f)(x)]^r &\leq C \sum_{k=j}^{\infty} 2^{(j-k)(Ar-n)} \left\{ M^{\text{loc}}(|(\psi_0)_k * f|^r)(x) \right. \\ &\quad \left. + K_{Br}(|(\psi_0)_k * f|^r)(x) \right\}, \end{aligned}$$

where

$$(\psi_0)_{j,A,B}^*(f)(x) \equiv \sup_{y \in \mathbb{R}^n} \frac{|(\psi_0)_j * f(x-y)|}{m_{j,A,B,x}(y)}$$

for all $x \in \mathbb{R}^n$.

Proof. First we can get the stronger version of (3.8) by virtue of (3.13), that is:

$$\begin{aligned} [(\psi_0)_{j,A,B}^*(f)(x)]^r &\leq C \sum_{k=j}^{\infty} 2^{(j-k)Ar} 2^{kn} \int_{\mathbb{R}^n} \frac{|(\psi_0)_k * f(y)|^r}{m_{j,Ar,Br,x}(x-y)} dy \\ &\leq C \sum_{k=j}^{\infty} 2^{(j-k)(Ar-n)} \left\{ 2^{jn} \int_{|y-x| < 2^{-j_x}} \frac{|(\psi_0)_k * f(y)|^r}{(1 + 2^j|x-y|)^{Ar}} dy \right. \\ &\quad \left. + 2^{jn} \int_{|y-x| \geq 2^{-j_x}} \frac{|(\psi_0)_k * f(y)|^r}{(2^j|x-y|)^{Ar} 2^{Br|x-y|/\rho(x)}} dy \right\} \\ &\equiv C \sum_{k=j}^{\infty} 2^{(j-k)(Ar-n)} \{I + II\}. \end{aligned}$$

Since $2^{-j_x} < \rho(x) \leq 2^{-j_x+1}$ and $j \geq j_x$, for I we have

$$\begin{aligned} I &= 2^{jn} \int_{2^{-j} \leq |y-x| < 2^{-j_x}} \frac{|(\psi_0)_k * f(y)|^r}{(1 + 2^j|x-y|)^{Ar}} dy + 2^{jn} \int_{|y-x| \leq 2^{-j}} \frac{|(\psi_0)_k * f(y)|^r}{(1 + 2^j|x-y|)^{Ar}} dy \\ &\equiv I_1 + I_2. \end{aligned}$$

According to the definition of $M^{loc}f(x)$ (see (2.3)), for I_2 we have

$$I_2 \leq 2^{jn} \int_{|y-x| \leq 2^{-j}} |(\psi_0)_k * f(y)|^r dy \leq CM^{loc}(|(\psi_0)_k * f|^r)(x),$$

and for I_1 we have

$$\begin{aligned} I_1 &\leq 2^{jn} \sum_{l=j_x+1}^j \int_{2^{-l} \leq |y-x| < 2^{-l+1}} \frac{|(\psi_0)_k * f(y)|^r}{(2^j|x-y|)^{Ar}} dy \\ &\leq \sum_{l=j_x+1}^j \frac{2^{jn}(2^{-l+1})^n}{(2^{j-l})^{Ar}} \frac{1}{(2^{-l+1})^n} \int_{|y-x| \leq 2^{-l+1}} |(\psi_0)_k * f(y)|^r dy \\ &\leq \sum_{l=j_x+1}^j \frac{2^n}{2^{(Ar-n)(j-l)}} M^{loc}(|(\psi_0)_k * f|^r)(x) \\ &\leq CM^{loc}(|(\psi_0)_k * f|^r)(x), \end{aligned}$$

where $Ar > n$. In addition, with regard to II , we have the following estimate,

$$\begin{aligned} II &\leq \frac{2^{jn}(\rho(x))^n}{(2^{j-j_x})^{Ar}} \frac{1}{(\rho(x))^n} \int_{\mathbb{R}^n} |(\psi_0)_k * f(y)|^r 2^{-Br \frac{|x-y|}{\rho(x)}} dy \\ &\leq C \frac{2^{jn}(2^{-j_x})^n}{(2^{j-j_x})^{Ar}} K_{Br}(|(\psi_0)_k * f|)(x) \\ &\leq C \frac{(2^{j-j_x})^n}{(2^{j-j_x})^{Ar}} K_{Br}(|(\psi_0)_k * f|)(x) \\ &\leq CK_{Br}(|(\psi_0)_k * f|)(x), \end{aligned}$$

where the last inequality is a consequence of the fact that $j \geq j_x$ and $Ar > n$. This finishes the proof. \square

Now we can establish weighted norm inequalities of $\psi_0^+(f)$, $\psi_{0,A,B}^{**}(f)$ and $\widetilde{\mathcal{M}}_{N,R}(f)$.

Theorem 3.1. *Let $\omega \in A_{\infty}^{\rho,\infty}(\mathbb{R}^n)$, $R \in (0, \infty)$, $p \in (0, 1]$, ψ_0 and q_ω be respectively as in (3.3) and (2.4), and let $\psi_0^+(f)$, $\psi_{0,A,B}^{**}(f)$ and $\widetilde{\mathcal{M}}_{N,R}(f)$ be respectively as in (3.4), (3.5) and (3.1). Let $A_1 \equiv \max\{A_0, nq_\omega/p\}$, $B_1 \equiv B_0/p$ and $N_0 \equiv [2A_1] + 1$, where A_0 and B_0 are respectively as in Lemmas 3.2 and 3.3. Then for any $A \in (A_1, \infty)$, $B \in (B_1, \infty)$ and integer $N \geq N_0$, there exists a positive constant C , depending only on $A, B, N, R, \psi_0, \omega$ and n , such that for all $f \in \mathcal{D}'(\mathbb{R}^n)$,*

$$\|\psi_{0,A,B}^{**}(f)\|_{L_{\omega}^p(\mathbb{R}^n)} \leq C \|\psi_0^+(f)\|_{L_{\omega}^p(\mathbb{R}^n)}, \quad (3.16)$$

and

$$\left\| \widetilde{\mathcal{M}}_{N,R}(f) \right\|_{L^p_\omega(\mathbb{R}^n)} \leq C \left\| \psi_0^+(f) \right\|_{L^p_\omega(\mathbb{R}^n)}, \quad (3.17)$$

Proof. Let $f \in \mathcal{D}'(\mathbb{R}^n)$. First, we prove (3.16). Let $A \in (A_1, \infty)$ and $B \in (B_1, \infty)$. By $A_1 \equiv \max\{A_0, nq_\omega/p\}$ and $B_1 \equiv B_0/p$, we know that there exists $r_0 \in (0, p/q_\omega)$ such that $A > n/r_0$ and $Br_0 > B_0/q_\omega$, where A_0 and B_0 are respectively as in Lemmas 3.2 and 3.3. Thus, by Lemma 3.4, for all $x \in \mathbb{R}^n$ and $j \geq j_x$ we have

$$\begin{aligned} [(\psi_0)_{j,A,B}^*(f)(x)]^{r_0} &\lesssim \sum_{k=j}^{\infty} 2^{(j-k)(Ar_0-n)} \left\{ M^{\text{loc}}(|(\psi_0)_k * f|^{r_0})(x) \right. \\ &\quad \left. + K_{Br_0}(|(\psi_0)_k * f|^{r_0})(x) \right\}. \end{aligned} \quad (3.18)$$

Let $\psi_0^+(f)$ and $\psi_{0,A,B}^{**}(f)$ be respectively as in (3.4) and (3.5). We notice that for any $x \in \mathbb{R}^n$ and $k \geq j_x$,

$$|(\psi_0)_k * f(x)| \leq \psi_0^+(f)(x),$$

which together with (3.18) implies that for all $x \in \mathbb{R}^n$,

$$[\psi_{0,A,B}^{**}(f)(x)]^{r_0} \lesssim M^{\text{loc}}([\psi_0^+(f)]^{r_0})(x) + K_{Br_0}([\psi_0^+(f)]^{r_0})(x). \quad (3.19)$$

Then by (3.19) we have

$$\begin{aligned} \int_{\mathbb{R}^n} |\psi_{0,A,B}^{**}(f)(x)|^p \omega(x) dx &\lesssim \int_{\mathbb{R}^n} \left| \left\{ M^{\text{loc}}([\psi_0^+(f)]^{r_0})(x) \right\} \right|^{p/r_0} \omega(x) dx \\ &\quad + \int_{\mathbb{R}^n} \left| \left\{ K_{Br_0}([\psi_0^+(f)]^{r_0})(x) \right\} \right|^{p/r_0} \omega(x) dx \\ &\equiv I_1 + I_2. \end{aligned} \quad (3.20)$$

For I_1 , as $r_0 < p/q_\omega$, we have $q \equiv p/r_0 > q_\omega$ and $\omega \in A_q^{\rho,\infty}(\mathbb{R}^n)$, therefore by Lemma 2.4(vii) we get

$$\int_{\mathbb{R}^n} \left| M^{\text{loc}}([\psi_0^+(f)]^{r_0})(x) \right|^{p/r_0} \omega(x) dx \lesssim \int_{\mathbb{R}^n} |\psi_0^+(f)|^p \omega(x) dx \quad (3.21)$$

and for I_2 by Lemma 3.3 we get

$$\int_{\mathbb{R}^n} |K_{Br_0}([\psi_0^+(f)]^{r_0})(x)|^{p/r_0} \omega(x) dx \lesssim \int_{\mathbb{R}^n} |\psi_0^+(f)|^p \omega(x) dx, \quad (3.22)$$

which together with (3.21) implies (3.16).

Now we prove (3.17). By $N_0 \equiv [2A_1] + 1$, we know that there exists $A \in (A_1, \infty)$ such that $2A < N_0$. In the rest of this proof, we fix $A \in (A_1, \infty)$ satisfying $2A < N_0$ and $B \in (B_1, \infty)$. Take an integer $N \geq N_0$ and $R \in (0, \infty)$. For any $\gamma \in \mathcal{D}_{N,R}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $l \in \mathbb{Z}$ (where l

satisfies $2^{-l} \in (0, \rho(x))$ and $j \geq j_x$ (where $2^{-j_x} < \rho(x) \leq 2^{-j_x+1}$), from Lemma 3.1, it follows that

$$\gamma_l * f = \gamma_l * (\varphi_0)_j * (\psi_0)_j * f + \sum_{k=j+1}^{\infty} \gamma_l * \varphi_k * \psi_k * f, \quad (3.23)$$

where $\varphi_0, \varphi \in \mathcal{D}(\mathbb{R}^n)$ with $L_\varphi \geq N$ and ψ is as in Lemma 3.1.

For any given $l_0 \in \mathbb{Z}$ which satisfies $2^{-l_0} \in (0, \rho(x))$, and $z \in \mathbb{R}^n$ which satisfies $|z - x| < 2^{-l_0}$, by (3.23) we have

$$\begin{aligned} |\gamma_{l_0} * f(z)| &\leq |\gamma_{l_0} * (\varphi_0)_{l_0} * (\psi_0)_{l_0} * f(z)| + \sum_{k=l_0+1}^{\infty} |\gamma_{l_0} * \varphi_k * \psi_k * f(z)| \\ &\leq \int_{\mathbb{R}^n} |\gamma_{l_0} * (\varphi_0)_{l_0}(y)| |(\psi_0)_{l_0} * f(z - y)| dy \\ &\quad + \sum_{k=l_0+1}^{\infty} \int_{\mathbb{R}^n} |\gamma_{l_0} * \varphi_k(y)| |\psi_k * f(z - y)| dy \equiv I_3 + I_4. \end{aligned} \quad (3.24)$$

To estimate I_3 , from

$$\begin{aligned} \psi_{0,A,B}^{**}(f)(x) &= \sup_{j \geq j_x, y \in \mathbb{R}^n} \frac{|(\psi_0)_j * f(x - y)|}{m_{j,A,B,x}(y)} \\ &= \sup_{j \geq j_x, y \in \mathbb{R}^n} \frac{|(\psi_0)_j * f(x - (y + x - z))|}{m_{j,A,B,x}(y + x - z)} \\ &= \sup_{j \geq j_x, y \in \mathbb{R}^n} \frac{|(\psi_0)_j * f(z - y)|}{m_{j,A,B,x}(y + x - z)}, \end{aligned}$$

we infer that

$$|(\psi_0)_{l_0} * f(z - y)| \leq \psi_{0,A,B}^{**}(f)(x) m_{l_0,A,B,x}(y + x - z),$$

which together with the facts that

$$m_{l_0,A,B,x}(y + x - z) \leq m_{l_0,A,B,x}(x - z) m_{l_0,A,B,x}(y)$$

and

$$m_{l_0,A,B,x}(x - z) = (1 + 2^{l_0}|x - z|)^A 2^{B \frac{|x-z|}{\rho(x)}} \lesssim 2^A,$$

implies that

$$|(\psi_0)_{l_0} * f(z - y)| \lesssim 2^A \psi_{0,A,B}^{**}(f)(x) m_{l_0,A,B,x}(y).$$

Thus, we have

$$I_3 \lesssim 2^A \left\{ \int_{\mathbb{R}^n} |\gamma_{l_0} * (\varphi_0)_{l_0}(y)| m_{l_0,A,B,x}(y) dy \right\} \psi_{0,A,B}^{**}(f)(x).$$

To estimate I_4 , by the definition of ψ , it is easy to know that for any $k \in \mathbb{Z}$,

$$|\psi_k * f(z - y)| \leq |(\psi_0)_k * f(z - y)| + |(\psi_0)_{k-1} * f(z - y)|.$$

By the definition of $\psi_{0,A,B}^{**}(f)$ and the facts that

$$m_{k,A,B,x}(y+x-z) \leq m_{k,A,B,x}(x-z)m_{k,A,B,x}(y),$$

for any $k \in \mathbb{Z}$ and $m_{k,A,B,x}(x-z) \lesssim 2^{(k-l_0)A}$, we conclude that

$$\begin{aligned} |(\psi_0)_k * f(z-y)| &\leq \psi_{0,A,B}^{**}(f)(x)m_{k,A,B,x}(y+x-z) \\ &\leq \psi_{0,A,B}^{**}(f)(x)m_{k,A,B,x}(x-z)m_{k,A,B,x}(y) \\ &\lesssim 2^{(k-l_0)A}m_{k,A,B,x}(y)\psi_{0,A,B}^{**}(f)(x). \end{aligned}$$

Similarly, we also have

$$|(\psi_0)_{k-1} * f(z-y)| \lesssim 2^{(k-l_0)A}m_{k,A,B,x}(y)\psi_{0,A,B}^{**}(f)(x).$$

Thus,

$$I_4 \lesssim \sum_{k=l_0+1}^{\infty} 2^{(k-l_0)A} \left\{ \int_{\mathbb{R}^n} |\gamma_t * \varphi_k(y)| m_{k,A,B,x}(y) dy \right\} \psi_{0,A,B}^{**}(f)(x).$$

From (3.24) and the above estimates of I_3 and I_4 , it follows that

$$\begin{aligned} |\gamma_{l_0} * f(z)| &\lesssim \left\{ \int_{\mathbb{R}^n} |\gamma_{l_0} * (\varphi_0)_{l_0}(y)| m_{l_0,A,B,x}(y) dy \right. \\ &\quad \left. + \sum_{k=l_0+1}^{\infty} 2^{(k-l_0)A} \int_{\mathbb{R}^n} |\gamma_{l_0} * \varphi_k(y)| m_{k,A,B,x}(y) dy \right\} \psi_{0,A,B}^{**}(f)(x). \end{aligned} \quad (3.25)$$

Assume that $\text{supp}(\varphi_0) \subset B(0, R_0)$. Then $\text{supp}((\varphi_0)_j) \subset B(0, 2^{-j}R_0)$ for all $j \geq j_x$. Moreover, by $\text{supp}(\gamma) \subset B(0, R)$, we see that $\text{supp}(\gamma_{l_0}) \subset B(0, 2^{-l_0}R)$. From this, we further deduce that $\text{supp}(\gamma_{l_0} * (\varphi_0)_{l_0}) \subset B(0, 2^{-l_0}(R_0 + R))$ and

$$|\gamma_{l_0} * (\varphi_0)_{l_0}(y)| \lesssim \int_{\mathbb{R}^n} |\gamma_{l_0}(s)| |(\varphi_0)_{l_0}(y-s)| ds \lesssim 2^{l_0n} \int_{\mathbb{R}^n} |\gamma_{l_0}(s)| ds \sim 2^{l_0n},$$

which implies that

$$\int_{\mathbb{R}^n} |\gamma_{l_0} * (\varphi_0)_{l_0}(y)| m_{l_0,A,B,x}(y) dy \lesssim 2^{l_0n} \int_{B(0, 2^{-l_0}(R_0+R))} (1 + 2^{l_0}|y|)^A 2^{\frac{B|y|}{\rho(x)}} dy \lesssim 1. \quad (3.26)$$

Moreover, since φ has vanishing moments up to order N , it was proved in [23, (2.13)] that

$$\|\gamma_{l_0} * \varphi_k\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^{(l_0-k)N} 2^{l_0n}$$

for all $k \in \mathbb{Z}$ with $k \geq l_0 + 1$, which, together with the facts that $l_0 \geq j_x$, $N > 2A$ and

$$\text{supp}(\gamma_{l_0} * \varphi_k) \subset B(0, 2^{-l_0}R_0 + 2^{-k}R),$$

implies that

$$\begin{aligned}
& \sum_{k=l_0+1}^{\infty} 2^{(k-l_0)A} \int_{\mathbb{R}^n} |\gamma_{l_0} * \varphi_k(y)| m_{k,A,B,x}(y) dy \\
& \lesssim \sum_{k=l_0+1}^{\infty} 2^{(k-l_0)A} 2^{(l_0-k)N} 2^{l_0 n} (2^{-l_0} R_0 + 2^{-k} R)^n \\
& \quad \times \left[1 + 2^k (2^{-l_0} R_0 + 2^{-k} R) \right]^A 2^{B(2^{-l_0} R_0 + 2^{-k} R)/\rho(x)} \\
& \lesssim \sum_{k=l_0+1}^{\infty} 2^{(l_0-k)(N-2A)} \lesssim 1.
\end{aligned} \tag{3.27}$$

Thus, from (3.25), (3.26) and (3.27), we deduce that $|\gamma_{l_0} * f(z)| \lesssim \psi_{0,A,B}^{**}(f)(x)$. Then, by the arbitrariness of $l_0 \geq j_x$ and $z \in B(x, 2^{-l_0})$, we know that

$$\widetilde{\mathcal{M}}_{N,R}(f)(x) \lesssim \psi_{0,A,B}^{**}(f)(x), \tag{3.28}$$

which deduces the (3.17) and completes the proof of this theorem. \square

As a corollary of Theorem 3.1, we immediately obtain the local vertical and the local nontangential maximal function characterizations of $h_{\rho,N}^p(\omega)$ with $N \geq N_{p,\omega}$ as follows. Here and in what follows,

$$N_{p,\omega} \equiv \max \left\{ \widetilde{N}_{p,\omega}, N_0 \right\}, \tag{3.29}$$

where $\widetilde{N}_{p,\omega}$ and N_0 are respectively as in Definition 3.2 and Theorem 3.1.

Theorem 3.2. *Let $\omega \in A_{\infty}^{\rho,\infty}(\mathbb{R}^n)$, ψ_0 and $N_{p,\omega}$ be respectively as in (3.3) and (3.29). Then for any integer $N \geq N_{p,\omega}$, the following are equivalent:*

- (i) $f \in h_{\rho,N}^p(\omega)$;
- (ii) $f \in \mathcal{D}'(\mathbb{R}^n)$ and $\psi_0^+(f) \in L_{\omega}^p(\mathbb{R}^n)$;
- (iii) $f \in \mathcal{D}'(\mathbb{R}^n)$ and $(\psi_0)_{\nabla}^*(f) \in L_{\omega}^p(\mathbb{R}^n)$;
- (iv) $f \in \mathcal{D}'(\mathbb{R}^n)$ and $\widetilde{\mathcal{M}}_N(f) \in L_{\omega}^p(\mathbb{R}^n)$;
- (v) $f \in \mathcal{D}'(\mathbb{R}^n)$ and $\widetilde{\mathcal{M}}_N^0(f) \in L_{\omega}^p(\mathbb{R}^n)$;
- (vi) $f \in \mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{M}_N^0(f) \in L_{\omega}^p(\mathbb{R}^n)$.

Moreover, for all $f \in h_{\rho,N}^p(\omega)$

$$\begin{aligned}
\|f\|_{h_{\rho,N}^p(\omega)} & \sim \|\psi_0^+(f)\|_{L_{\omega}^p(\mathbb{R}^n)} \sim \|(\psi_0)_{\nabla}^*(f)\|_{L_{\omega}^p(\mathbb{R}^n)} \\
& \sim \|\widetilde{\mathcal{M}}_N(f)\|_{L_{\omega}^p(\mathbb{R}^n)} \sim \|\widetilde{\mathcal{M}}_N^0(f)\|_{L_{\omega}^p(\mathbb{R}^n)} \sim \|\mathcal{M}_N^0(f)\|_{L_{\omega}^p(\mathbb{R}^n)},
\end{aligned} \tag{3.30}$$

where the implicit constants are independent of f .

Proof. (i) \Rightarrow (ii). Pick an integer $N \geq N_{p,\omega}$ and $f \in h_{\rho,N}^p(\omega)$. Let $\tilde{\psi}_0$ satisfy (3.3) and $\tilde{\psi}_0 \in \mathcal{D}_N(\mathbb{R}^n)$. Then from the definition of $\mathcal{M}_N(f)$, we infer that $\tilde{\psi}_0^+(f) \leq \mathcal{M}_N(f)$ and hence $\tilde{\psi}_0^+(f) \in L_\omega^p(\mathbb{R}^n)$. For any ψ_0 satisfying (3.3), assume that $\text{supp}(\psi_0) \subset B(0, R)$. Then, by (3.17) and the above argument, we have

$$\left\| \widetilde{\mathcal{M}}_{N,R}(f) \right\|_{L_\omega^p(\mathbb{R}^n)} \lesssim \left\| \tilde{\psi}_0^+(f) \right\|_{L_\omega^p(\mathbb{R}^n)} \lesssim \|f\|_{h_{\rho,N}^p(\omega)},$$

which together with $\psi_0^+(f) \lesssim \widetilde{\mathcal{M}}_{N,R}(f)$ implies that $\psi_0^+(f) \in L_\omega^p(\mathbb{R}^n)$ and

$$\left\| \psi_0^+(f) \right\|_{L_\omega^p(\mathbb{R}^n)} \lesssim \|f\|_{h_{\rho,N}^p(\omega)}.$$

(ii) \Rightarrow (iii). Let $f \in \mathcal{D}'(\mathbb{R}^n)$ satisfy $\psi_0^+(f) \in L_\omega^p(\mathbb{R}^n)$, where ψ_0 is as in (3.3). Then from the fact that

$$\psi_0^+(f) \leq (\psi_0)_\nabla^*(f) \lesssim \psi_{0,A,B}^{*,*}(f)$$

and (3.16), we deduce that $(\psi_0)_\nabla^*(f) \in L_\omega^p(\mathbb{R}^n)$ and

$$\|(\psi_0)_\nabla^*(f)\|_{L_\omega^p(\mathbb{R}^n)} \lesssim \|\psi_0^+(f)\|_{L_\omega^p(\mathbb{R}^n)}.$$

(iii) \Rightarrow (iv). Let $f \in \mathcal{D}'(\mathbb{R}^n)$ satisfy $(\psi_0)_\nabla^*(f) \in L_\omega^p(\mathbb{R}^n)$, where ψ_0 is as in (3.3). By (3.17),

$$\left\| \widetilde{\mathcal{M}}_N(f) \right\|_{L_\omega^p(\mathbb{R}^n)} \lesssim \|\psi_0^+(f)\|_{L_\omega^p(\mathbb{R}^n)},$$

which together with the fact that

$$\psi_0^+(f) \leq (\psi_0)_\nabla^*(f)$$

and the assumption that $(\psi_0)_\nabla^*(f) \in L_\omega^p(\mathbb{R}^n)$ implies $\widetilde{\mathcal{M}}_N(f) \in L_\omega^p(\mathbb{R}^n)$ and

$$\left\| \widetilde{\mathcal{M}}_N(f) \right\|_{L_\omega^p(\mathbb{R}^n)} \lesssim \|(\psi_0)_\nabla^*(f)\|_{L_\omega^p(\mathbb{R}^n)}.$$

(iv) \Rightarrow (v) \Rightarrow (vi). By the facts that $\mathcal{M}_N^0(f) \leq \widetilde{\mathcal{M}}_N^0(f) \leq \widetilde{\mathcal{M}}_N(f)$ for any $f \in \mathcal{D}'(\mathbb{R}^n)$, we see that all the conclusions hold. Moreover, it is obvious that

$$\left\| \mathcal{M}_N^0(f) \right\|_{L_\omega^p(\mathbb{R}^n)} \leq \left\| \widetilde{\mathcal{M}}_N^0(f) \right\|_{L_\omega^p(\mathbb{R}^n)} \leq \left\| \widetilde{\mathcal{M}}_N(f) \right\|_{L_\omega^p(\mathbb{R}^n)}.$$

(vi) \Rightarrow (i). Let $f \in \mathcal{D}'(\mathbb{R}^n)$ satisfy $\mathcal{M}_N^0(f) \in L_\omega^p(\mathbb{R}^n)$. Let ψ_1 satisfy (3.3) and $\psi_1 \in \mathcal{D}_N^0(\mathbb{R}^n)$. Then by (3.17), we have that

$$\left\| \widetilde{\mathcal{M}}_N(f) \right\|_{L_\omega^p(\mathbb{R}^n)} \lesssim \|\psi_1^+(f)\|_{L_\omega^p(\mathbb{R}^n)},$$

which together with the facts that $\psi_1^+(f) \leq \mathcal{M}_N^0(f)$ and $\mathcal{M}_N(f) \leq \widetilde{\mathcal{M}}_N(f)$ implies that

$$\left\| \mathcal{M}_N(f) \right\|_{L_\omega^p(\mathbb{R}^n)} \lesssim \left\| \mathcal{M}_N^0(f) \right\|_{L_\omega^p(\mathbb{R}^n)}.$$

Thus, by the definition of $h_{\rho,N}^p(\omega)$, we know that $f \in h_{\rho,N}^p(\omega)$ and

$$\|f\|_{h_{\rho,N}^p(\omega)} \lesssim \|\mathcal{M}_N^0(f)\|_{L_\omega^p(\mathbb{R}^n)},$$

which completes the proof of Theorem 3.2. \square

By the Theorems 3.1 and 3.2, we also have the following corollary about local tangential maximal function characterization of $h_{\rho,N}^p(\omega)$, and we omit the details here.

Corollary 3.1. *Let ψ_0 be as in (3.3), $\omega \in A_\infty^{\rho,\infty}(\mathbb{R}^n)$, $N_{p,\omega}$ be as in (3.29), A and B be as in Theorem 3.1. Then for integer $N \geq N_{p,\omega}$, $f \in h_{\rho,N}^p(\omega)$ if and only if $f \in \mathcal{D}'(\mathbb{R}^n)$ and $\psi_{0,A,B}^{**}(f) \in L_\omega^p(\mathbb{R}^n)$; moreover,*

$$\|f\|_{h_{\rho,N}^p(\omega)} \sim \|\psi_{0,A,B}^{**}(f)\|_{L_\omega^p(\mathbb{R}^n)}.$$

Next we give some basic properties of $h_{\rho,N}^p(\omega)$ and $h_{\rho}^{p,q,s}(\omega)$.

Proposition 3.2. *Let $\omega \in A_\infty^{\rho,\infty}(\mathbb{R}^n)$, $p \in (0, 1]$ and $N_{p,\omega}$ be as in (3.29). For any integer $N \geq N_{p,\omega}$, the inclusion $h_{\rho,N}^p(\omega) \hookrightarrow \mathcal{D}'(\mathbb{R}^n)$ is continuous.*

Proof. First, for any $x \in B(0, \rho(0))$, by Lemma 2.1, there exist $C_0 \geq 1$ and $k_0 \geq 1$, such that

$$\rho(0) \leq C_0 \left(1 + \frac{|x|}{\rho(0)}\right)^{k_0} \rho(x) \leq C_0 2^{k_0} \rho(x).$$

We take $r_1 \equiv \rho(0)/C_0 2^{k_0+1} < \min\{\rho(x), \rho(0)\}$, then we have $B(0, r_1) \subset B(0, \rho(0))$. In addition, for any $x \in B(0, r_1)$, we also have $|x| < r_1 < \rho(x)$.

Next, let $f \in h_{\rho,N}^p(\omega)$. For any given $\varphi \in \mathcal{D}(\mathbb{R}^n)$, assume that $\text{supp}(\varphi) \subset B(0, R)$ with $R \in (0, \infty)$. Then by Theorem 3.1 and 3.2, we have

$$\begin{aligned} |\langle f, \varphi \rangle| &= |f * \tilde{\varphi}(0)| \leq \|\tilde{\varphi}\|_{\mathcal{D}_{N,R}(\mathbb{R}^n)} \inf_{x \in B(0, r_1)} \widetilde{\mathcal{M}}_{N,R}(f)(x) \\ &\leq \|\tilde{\varphi}\|_{\mathcal{D}_{N,R}(\mathbb{R}^n)} [\omega(B(0, r_1))]^{-1/p} \left\| \widetilde{\mathcal{M}}_{N,R}(f) \right\|_{L_\omega^p(\mathbb{R}^n)} \\ &\lesssim \|\tilde{\varphi}\|_{\mathcal{D}_{N,R}(\mathbb{R}^n)} [\omega(B(0, r_1))]^{-1/p} \|f\|_{h_{\rho,N}^p(\omega)}, \end{aligned}$$

where $\widetilde{\mathcal{M}}_{N,R}(f)$ is as in (3.1) and $\tilde{\varphi}(x) \equiv \varphi(-x)$ for all $x \in \mathbb{R}^n$. This implies $f \in \mathcal{D}'(\mathbb{R}^n)$ and the inclusion is continuous. The proof is finished. \square

Proposition 3.3. *Let $\omega \in A_\infty^{\rho,\infty}(\mathbb{R}^n)$, $p \in (0, 1]$ and $N_{p,\omega}$ be as in (3.29). For any integer $N \geq N_{p,\omega}$, the space $h_{\rho,N}^p(\omega)$ is complete.*

Proof. For any $\psi \in \mathcal{D}_N(\mathbb{R}^n)$ and $\{f_i\}_{i \in \mathbb{N}} \subset \mathcal{D}'(\mathbb{R}^n)$ such that $\{\sum_{i=1}^j f_i\}_{j \in \mathbb{N}}$ converges in $\mathcal{D}'(\mathbb{R}^n)$ to a distribution f as $j \rightarrow \infty$, and the series $\sum_i f_i * \psi(x)$ converges pointwise to $f * \psi(x)$ for each $x \in \mathbb{R}^n$. Therefore,

$$(\mathcal{M}_N(f)(x))^p \leq \left(\sum_{i=1}^{\infty} \mathcal{M}_N(f_i)(x) \right)^p \leq \sum_{i=1}^{\infty} (\mathcal{M}_N(f_i)(x))^p \quad \text{for all } x \in \mathbb{R}^n,$$

and hence $\|f\|_{h_{\rho,N}^p(\omega)} \leq \sum_i \|f_i\|_{h_{\rho,N}^p(\omega)}$.

To prove that $h_{\rho,N}^p(\omega)$ is complete, it suffices to show that for every sequence $\{f_j\}_{j \in \mathbb{N}}$ with $\|f_j\|_{h_{\rho,N}^p(\omega)} < 2^{-j}$ for any $j \in \mathbb{N}$, the series $\sum_{j \in \mathbb{N}} f_j$ convergence in $h_{\rho,N}^p(\omega)$. Since $\{\sum_{i=1}^j f_i\}_{j \in \mathbb{N}}$ is a Cauchy sequence in $h_{\rho,N}^p(\omega)$, by Proposition 3.2 and the completeness of $\mathcal{D}'(\mathbb{R}^n)$, $\{\sum_{i=1}^j f_i\}_{j \in \mathbb{N}}$ is also a Cauchy sequence in $\mathcal{D}'(\mathbb{R}^n)$ and thus converges to some $f \in \mathcal{D}'(\mathbb{R}^n)$. Therefore,

$$\left\| f - \sum_{i=1}^j f_i \right\|_{h_{\rho,N}^p(\omega)}^p = \left\| \sum_{i=j+1}^{\infty} f_i \right\|_{h_{\rho,N}^p(\omega)}^p \leq \sum_{i=j+1}^{\infty} 2^{-ip} \rightarrow 0$$

as $j \rightarrow \infty$. This finishes the proof. \square

Theorem 3.3. *Let $\omega \in A_{\infty}^{\rho,\infty}(\mathbb{R}^n)$ and $N_{p,\omega}$ be as in (3.29). If $(p, q, s)_{\omega}$ is an admissible triplet (see Definition 3.3) and integer $N \geq N_{p,\omega}$, then*

$$h_{\rho}^{p,q,s}(\omega) \subset h_{\rho,N_{p,\omega}}^p(\omega) \subset h_{\rho,N}^p(\omega),$$

and moreover, there exists a positive constant C such that for all $f \in h_{\rho}^{p,q,s}(\omega)$,

$$\|f\|_{h_{\rho,N}^p(\omega)} \leq \|f\|_{h_{\rho,N_{p,\omega}}^p(\omega)} \leq C \|f\|_{h_{\rho}^{p,q,s}(\omega)}.$$

Proof. Obviously, we only need to prove $h_{\rho}^{p,q,s}(\omega) \subset h_{\rho,N_{p,\omega}}^p(\omega)$. For all $f \in h_{\rho}^{p,q,s}(\omega)$,

$$\|f\|_{h_{\rho,N_{p,\omega}}^p(\omega)} \lesssim \|f\|_{h_{\rho}^{p,q,s}(\omega)}.$$

By Definition 3.4 and Theorem 3.2, it suffices to prove that there exists a positive constant C such that

$$\left\| \mathcal{M}_{N_{p,\omega}}^0(a) \right\|_{L_{\omega}^p(\mathbb{R}^n)} \leq C, \quad \text{for all } (p, q)_{\omega} - \text{single-atoms } a, \quad (3.31)$$

and

$$\left\| \mathcal{M}_{N_{p,\omega}}^0(a) \right\|_{L_{\omega}^p(\mathbb{R}^n)} \leq C, \quad \text{for all } (p, q, s)_{\omega} - \text{atoms } a. \quad (3.32)$$

We first prove (3.31). Since $q \in (q_{\omega}, \infty]$, so $\omega \in A_q^{\rho,\infty}(\mathbb{R}^n)$. Let a be a $(p, q)_{\omega}$ -single-atom. When $\omega(\mathbb{R}^n) = \infty$, by the definition of the single atom, we know that $a = 0$ for almost every

$x \in \mathbb{R}^n$. In this case, it is easy to see that (3.31) holds. When $\omega(\mathbb{R}^n) < \infty$, from Hölder's inequality, $\omega \in A_q^{\rho, \infty}(\mathbb{R}^n)$ and Proposition 3.1(i), we deduce that

$$\begin{aligned} \left\| \mathcal{M}_{N_p, \omega}^0(a) \right\|_{L_\omega^p(\mathbb{R}^n)}^p &= \int_{\mathbb{R}^n} \left| \mathcal{M}_{N_p, \omega}^0(a)(x) \right|^p \omega(x) dx \\ &\leq \left(\int_{\mathbb{R}^n} \left| \mathcal{M}_{N_p, \omega}^0(a)(x) \right|^q \omega(x) dx \right)^{p/q} \left(\int_{\mathbb{R}^n} \omega(x) dx \right)^{1-p/q} \\ &\leq C \|a\|_{L_\omega^q(\mathbb{R}^n)}^p [\omega(\mathbb{R}^n)]^{1-p/q} \leq C. \end{aligned}$$

Next, we prove (3.32). Let a be a $(p, q, s)_\omega$ -atom supported in the cube $Q \equiv Q(x_0, r)$. We consider the following two cases for Q .

The first case is when $r < L_2 \rho(x_0)$. Let $\tilde{Q} \equiv 2\sqrt{n}Q$, then we have

$$\begin{aligned} \int_{\mathbb{R}^n} \left| \mathcal{M}_{N_p, \omega}^0(a)(x) \right|^p \omega(x) dx &= \int_{\tilde{Q}} \left| \mathcal{M}_{N_p, \omega}^0(a)(x) \right|^p \omega(x) dx + \int_{\tilde{Q}^c} \left| \mathcal{M}_{N_p, \omega}^0(a)(x) \right|^p \omega(x) dx \\ &\equiv I_1 + I_2. \end{aligned} \quad (3.33)$$

For I_1 , by Hölder's inequality and the properties of $A_q^{\rho, \theta}(\mathbb{R}^n)$ (see Lemma 2.4(vi)), we have

$$\begin{aligned} I_1 &\leq \left(\int_{\mathbb{R}^n} \left| \mathcal{M}_{N_p, \omega}^0(a)(x) \right|^q \omega(x) dx \right)^{p/q} \left(\int_{\tilde{Q}} \omega(x) dx \right)^{1-p/q} \\ &\leq C \|a\|_{L_\omega^q(\mathbb{R}^n)}^p [\omega(\tilde{Q})]^{1-p/q} \leq C. \end{aligned} \quad (3.34)$$

To estimate I_2 , we claim that for $x \in \tilde{Q}^c$

$$\mathcal{M}_{N_p, \omega}^0(a)(x) \leq C |Q|^{(s_0+n+1)/n} [\omega(Q)]^{-1/p} |x - x_0|^{-(s_0+n+1)} \chi_{B(x_0, c_1 \rho(x_0))}(x), \quad (3.35)$$

where $s_0 \equiv [n(q_\omega/p - 1)]$ and $c_1 > 2\sqrt{n}$ is a constant independent of the atom a . Indeed, for any $\psi \in \mathcal{D}_N^0(\mathbb{R}^n)$ and $2^{-l} \in (0, \rho(x))$, let P be the Taylor expansion of ψ about $(x - x_0)/2^{-l}$ with degree s_0 . By Taylor's remainder theorem, for any $y \in \mathbb{R}^n$, we have

$$\begin{aligned} &\left| \psi \left(\frac{x - y}{2^{-l}} \right) - P \left(\frac{x - x_0}{2^{-l}} \right) \right| \\ &\leq C \sum_{\substack{\alpha \in \mathbb{Z}_+^n \\ |\alpha| = s_0 + 1}} \left| (\partial^\alpha \psi) \left(\frac{\theta(x - y) + (1 - \theta)(x - x_0)}{2^{-l}} \right) \right| \left| \frac{x_0 - y}{2^{-l}} \right|^{s_0 + 1}, \end{aligned}$$

where $\theta \in (0, 1)$. By $2^{-l} \in (0, \rho(x))$ and $x \in \tilde{Q}^c$, we see that $\text{supp}(a * \psi_l) \subset B(x_0, c_1 \rho(x_0))$, and $a * \psi_l(x) \neq 0$ implies that $2^{-l} > |x - x_0|/2$. Thus, from the above facts and Definition

3.3, we get that for all $x \in \tilde{Q}^{\mathbb{C}}$,

$$\begin{aligned}
|a * \psi_l(x)| &\leq \frac{1}{2^{-ln}} \left\{ \int_Q |a(y)| \left| \psi \left(\frac{x-y}{2^{-l}} \right) - P \left(\frac{x-x_0}{2^{-l}} \right) \right| dy \right\} \chi_{B(x_0, c_1 \rho(x_0))}(x) \\
&\leq C |x - x_0|^{-(s_0+n+1)} \left\{ \int_Q |a(y)| |x_0 - y|^{s_0+1} dy \right\} \chi_{B(x_0, c_1 \rho(x_0))}(x) \\
&\leq C |Q|^{(s_0+1)/n} \|a\|_{L_{\omega}^q(\mathbb{R}^n)} \left(\int_Q [\omega(y)]^{-q'/q} dy \right)^{1/q'} |x - x_0|^{-(s_0+n+1)} \chi_{B(x_0, c_1 \rho(x_0))}(x) \\
&\leq C |Q|^{(s_0+n+1)/n} [\omega(Q)]^{-1/p} |x - x_0|^{-(s_0+n+1)} \chi_{B(x_0, c_1 \rho(x_0))}(x),
\end{aligned}$$

which together with the arbitrariness of $\psi \in \mathcal{D}_N^0(\mathbb{R}^n)$ implies (3.35). Thus, the claim holds.

Let $Q_i \equiv 2^i \sqrt{n} Q$ for all $i \in \mathbb{N}$ and $i_0 \in \mathbb{N}$ satisfying $2^{i_0} r \leq c_1 \rho(x_0) < 2^{i_0+1} r$. As $s_0 = [n(q_\omega/p - 1)]$, we know that there exists $q_0 \in (q_\omega, \infty)$ such that $p(s_0 + n + 1) > nq_0$. Then from the Lemma 2.4, we conclude that

$$\begin{aligned}
I_2 &\leq \int_{\sqrt{n}r \leq |x-x_0| < c_1 \rho(x_0)} \left| \mathcal{M}_{N_p, \omega}^0(a)(x) \right|^p \omega(x) dx \\
&\leq C |Q|^{p(s_0+n+1)/n} [\omega(Q)]^{-1} \int_{\sqrt{n}r \leq |x-x_0| < c_1 \rho(x_0)} |x - x_0|^{-p(s_0+n+1)} \omega(x) dx \\
&\leq C r^{p(s_0+n+1)} [\omega(Q)]^{-1} \sum_{i=0}^{i_0} \int_{Q_{i+1} \setminus Q_i} |x - x_0|^{-p(s_0+n+1)} \omega(x) dx \\
&\leq C [\omega(Q)]^{-1} \sum_{i=0}^{i_0} 2^{-ip(s_0+n+1)} \omega(Q_{i+1}) \\
&\leq C [\omega(Q)]^{-1} \sum_{i=0}^{i_0} 2^{-i[p(s_0+n+1) - nq_0]} \omega(Q) \leq C,
\end{aligned}$$

which together with (3.33) and (3.34) implies (3.32) in the first case.

Now we consider the case $L_2 \rho(x_0) \leq r \leq L_1 \rho(x_0)$, let $Q^* \equiv Q(x_0, c_2 r)$, in which $c_2 > 1$ is an constant independent of atom a . Thus, from $\text{supp}(\mathcal{M}_{N_p, \omega}^0(a)) \subset Q^*$, Hölder's inequality and Lemma 2.4, we get

$$\begin{aligned}
\int_{\mathbb{R}^n} \left| \mathcal{M}_{N_p, \omega}^0(a)(x) \right|^p \omega(x) dx &= \int_{Q^*} \left| \mathcal{M}_{N_p, \omega}^0(a)(x) \right|^p \omega(x) dx \\
&\leq C \|a\|_{L_{\omega}^q(\mathbb{R}^n)}^p [\omega(Q^*)]^{1-p/q} \\
&\leq C \|a\|_{L_{\omega}^q(\mathbb{R}^n)}^p [\omega(Q)]^{1-p/q} \\
&\leq C.
\end{aligned}$$

This finishes the proof of Theorem 3.3. □

4 Calderón-Zygmund decompositions

In this section, we establish the Calderón-Zygmund decompositions associated with local grand maximal functions on weighted Euclidean space \mathbb{R}^n . We follow the constructions in [26], [1] and [5].

Let $\omega \in A_\infty^{\rho, \infty}(\mathbb{R}^n)$ and q_ω be as in (2.4). For integer $N \geq 2$, let $\mathcal{M}_N(f)$ and $\mathcal{M}_N^0(f)$ be as in (3.2). Throughout this section, we consider a distribution f so that for all $\lambda > 0$,

$$\omega(\{x \in \mathbb{R}^n : \mathcal{M}_N(f)(x) > \lambda\}) < \infty.$$

For a given $\lambda > \inf_{x \in \mathbb{R}^n} \mathcal{M}_N(f)(x)$, we set

$$\Omega_\lambda \equiv \{x \in \mathbb{R}^n : \mathcal{M}_N(f)(x) > \lambda\}.$$

It is obvious that Ω_λ is a proper open subset of \mathbb{R}^n . As in [26], we give the usual Whitney decomposition of Ω_λ . Thus we can find closed cubes Q_k whose interiors distance from Ω_λ^c , with $\Omega_\lambda = \bigcup_k Q_k$ and

$$\text{diam}(Q_k) \leq 2^{-(6+n)} \text{dist}(Q_k, \Omega_\lambda^c) \leq 4 \text{diam}(Q_k).$$

In what follows, fix $a \equiv 1 + 2^{-(11+n)}$ and $b \equiv 1 + 2^{-(10+n)}$, and if we denote $\bar{Q}_k = aQ_k$, $Q_k^* = bQ_k$, we have $Q_k \subset \bar{Q}_k \subset Q_k^*$. Moreover, $\Omega_\lambda = \bigcup_k Q_k^*$, and $\{Q_k^*\}_k$ have the bounded interior property, namely, every point in Ω_λ is contained in at most a fixed number of $\{Q_k^*\}_k$.

Now we take a function $\xi \in \mathcal{D}(\mathbb{R}^n)$ such that $0 \leq \xi \leq 1$, $\text{supp}(\xi) \subset aQ(0, 1)$ and $\xi \equiv 1$ on $Q(0, 1)$. For $x \in \mathbb{R}^n$, set $\xi_k(x) \equiv \xi((x - x_k)/l_k)$, where and in what follows, x_k is the center of the cube Q_k and l_k is its sidelength. Obviously, by the construction of $\{Q_k^*\}_k$ and $\{\xi_k\}_k$, for any $x \in \mathbb{R}^n$, we have $1 \leq \sum_k \xi_k(x) \leq M$, where M is a fixed positive integer independent of x . Let $\eta_k \equiv \xi_k / (\sum_j \xi_j)$. Then $\{\eta_k\}_k$ form a smooth partition of unity for Ω_λ subordinate to the locally finite covering $\{Q_k^*\}_k$ of Ω_λ , namely, $\chi_{\Omega_\lambda} = \sum_k \eta_k$ with each $\eta_k \in \mathcal{D}(\mathbb{R}^n)$ supported in \bar{Q}_k .

Let $s \in \mathbb{Z}_+$ be some fixed integer and $\mathcal{P}_s(\mathbb{R}^n)$ denote the linear space of polynomials in n variables of degrees no more than s . For each $i \in \mathbb{N}$ and $P \in \mathcal{P}_s(\mathbb{R}^n)$, set

$$\|P\|_i \equiv \left[\frac{1}{\int_{\mathbb{R}^n} \eta_i(y) dy} \int_{\mathbb{R}^n} |P(x)|^2 \eta_i(x) dx \right]^{1/2}. \quad (4.1)$$

Then it is easy to see that $(\mathcal{P}_s(\mathbb{R}^n), \|\cdot\|_i)$ is a finite dimensional Hilbert space. Let $f \in \mathcal{D}'(\mathbb{R}^n)$. Since f induces a linear functional on $\mathcal{P}_s(\mathbb{R}^n)$ via

$$P \mapsto \frac{1}{\int_{\mathbb{R}^n} \eta_i(y) dy} \langle f, P \eta_i \rangle,$$

by the Riesz represent theorem, there exists a unique polynomial $P_i \in \mathcal{P}_s(\mathbb{R}^n)$ for each i such that for all $Q \in \mathcal{P}_s(\mathbb{R}^n)$,

$$\langle f, Q\eta_i \rangle = \langle P_i, Q\eta_i \rangle = \int_{\mathbb{R}^n} P_i(x)Q(x)\eta_i(x) dx.$$

For each i , define the distribution $b_i \equiv (f - P_i)\eta_i$ when $l_i \in (0, L_3\rho(x_i))$ (where $L_3 = 2^{k_0}C_0$, x_i is the center of the cube Q_i) and $b_i \equiv f\eta_i$ when $l_i \in [L_3\rho(x_i), \infty)$.

We will show that for suitable choices of s and N , the series $\sum_i b_i$ converge in $\mathcal{D}'(\mathbb{R}^n)$, and in this case, we define $g \equiv f - \sum_i b_i$ in $\mathcal{D}'(\mathbb{R}^n)$. We point out that the represent $f = g + \sum_i b_i$, where g and b_i are as above, is called a Calderón-Zygmund decomposition of f of degree s and height λ associated with $\mathcal{M}_N(f)$.

The rest of this section consists of a series of lemmas. In Lemma 4.1 and Lemma 4.2, we give some properties of the smooth partition of unity $\{\eta_i\}_i$. In Lemmas 4.3 through 4.6, we derive some estimates for the bad parts $\{b_i\}_i$. Lemma 4.7 and Lemma 4.8 give controls over the good part g . Finally, Corollary 4.1 shows the density of $L_\omega^q(\mathbb{R}^n) \cap h_{\rho,N}^p(\omega)$ in $h_{\rho,N}^p(\omega)$, where $q \in (q_\omega, \infty)$.

Lemma 4.1. *There exists a positive constant C_1 depending only on N , such that for all i and $l \leq l_i$,*

$$\sup_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} |\partial^\alpha \eta_i(lx)| \leq C_1.$$

Lemma 4.1 is essentially Lemma 5.2 in [1].

Lemma 4.2. *If $l_i < L_3\rho(x_i)$, then there exists a constant $C_2 > 0$ independent of $f \in \mathcal{D}'(\mathbb{R}^n)$, l_i and $\lambda > 0$ so that*

$$\sup_{y \in \mathbb{R}^n} |P_i(y)\eta_i(y)| \leq C_2\lambda.$$

Proof. As in the proof of Lemma 5.3 in [1]. Let π_1, \dots, π_m ($m = \dim \mathcal{P}_s$) be an orthonormal basis of \mathcal{P}_s with respect to the norm (4.1). we have

$$P_i = \sum_{k=1}^m \left(\frac{1}{\int \eta_i} \int f(x)\pi_k(x)\eta_i(x) dx \right) \bar{\pi}_k, \quad (4.2)$$

where the integral is understood as $\langle f, \pi_k\eta_i \rangle$. Therefore,

$$\begin{aligned} 1 &= \frac{1}{\int \eta_i} \int_{\bar{Q}_i} |\pi_k(x)|^2 \eta_i(x) dx \geq \frac{2^{-n}}{|Q_i|} \int_{\bar{Q}_i} |\pi_k(x)|^2 \eta_i(x) dx \\ &\geq \frac{2^{-n}}{|Q_i|} \int_{Q_i} |\pi_k(x)|^2 dx = 2^{-n} \int_{Q^0} |\tilde{\pi}_k(x)|^2 dx, \end{aligned} \quad (4.3)$$

where $\tilde{\pi}_k(x) = \pi_k(x_i + l_i x)$ and Q^0 denotes the cube of side length 1 centered at the origin.

Since \mathcal{P}_s is finite dimensional, all norms on \mathcal{P}_s are equivalent, then there exists $A_1 > 0$ such that for all $P \in \mathcal{P}_s$

$$\sup_{|\alpha| \leq s} \sup_{z \in bQ^0} |\partial^\alpha P(z)| \leq A_1 \left(\int_{Q^0} |P(z)|^2 dz \right)^{1/2}.$$

From this and (4.3), for $k = 1, \dots, m$, we have

$$\sup_{|\alpha| \leq s} \sup_{z \in bQ^0} |\partial^\alpha \tilde{\pi}_k(z)| \leq A_1 2^{n/2}. \quad (4.4)$$

If $z \in 2^{8+n}nQ_i \cap \Omega^{\mathbb{L}}$, by Lemma 2.1, we have $\rho(x_i) \leq C_0(1 + 2^{8+n}n^2L_3)^{k_0}\rho(z)$, then we let $\tilde{L} \equiv 1/C_0(1 + 2^{8+n}n^2L_3)^{k_0}L_3$. For $k = 1, \dots, m$, we define

$$\Phi_k(y) = \frac{2^{-k_in}}{\int \eta_i} \pi_k(z - 2^{-k_i}y) \eta_i(z - 2^{-k_i}y),$$

where $z \in 2^{8+n}nQ_i \cap \Omega^{\mathbb{L}}$ and $2^{-k_i} \leq \tilde{L}l_i < 2^{-k_i+1}$. It is easy to see that $\text{supp} \Phi_k \subset B(0, R_1)$ where $R_1 \equiv 2^{9+n}n^2/\tilde{L}$, and $\|\Phi_k\|_{\mathcal{D}_N} \leq A_2$ by Lemma 4.1.

Note that

$$\frac{1}{\int \eta_i} \int f(x) \pi_k(x) \eta_i(x) dx = (f * (\Phi_k)_{k_i})(z),$$

since $2^{-k_i} \leq \tilde{L}l_i < \tilde{L}L_3\rho(x_i) \leq \rho(z)$, then we have

$$\left| \frac{1}{\int \eta_i} \int f(x) \pi_k(x) \eta_i(x) dx \right| \leq \mathcal{M}_N f(z) \|\Phi_k\|_{\mathcal{D}_N} \leq A_2 \lambda.$$

By (4.2), (4.4) and above estimate, we have

$$\sup_{z \in Q_i^*} |P_i(z)| \leq m 2^{n/2} A_1 A_2 \lambda.$$

Thus,

$$\sup_{z \in \mathbb{R}^n} |P_i(z) \eta_i(z)| \leq C_2 \lambda.$$

The proof is complete. \square

By the same method, we can get the following lemma as the Lemma 4.3 in [28], and we omit the details here.

Lemma 4.3. *There exists a constant $C_3 > 0$ such that*

$$\mathcal{M}_N^0 b_i(x) \leq C_3 \mathcal{M}_N f(x) \quad \text{for } x \in Q_i^*. \quad (4.5)$$

Lemma 4.4. Suppose $Q \subset \mathbb{R}^n$ is bounded, convex, and $0 \in Q$, and N is a positive integer. Then there is a constant C depending only on Q and N such that for every $\phi \in \mathcal{D}(\mathbb{R}^n)$ and every integer s , $0 \leq s < N$ we have

$$\sup_{z \in Q} \sup_{|\alpha| \leq N} |\partial^\alpha R_y(z)| \leq C \sup_{z \in y+Q} \sup_{s+1 \leq |\alpha| \leq N} |\partial^\alpha \phi(z)|,$$

where R_y is the remainder of the Taylor expansion of ϕ of order s at the point $y \in \mathbb{R}^n$.

Lemma 4.4 is Lemma 5.5 in [1].

Lemma 4.5. Suppose $0 \leq s < N$. Then there exist positive constants C_4, C_5 so that for $i \in \mathbb{N}$,

$$\mathcal{M}_N^0(b_i)(x) \leq C \frac{\lambda_i^{n+s+1}}{(l_i + |x - x_i|)^{n+s+1}} \chi_{\{|x-x_i| < C_4 \rho(x)\}}(x) \quad \text{if } x \notin Q_i^*. \quad (4.6)$$

Moreover,

$$\mathcal{M}_N^0(b_i)(x) = 0, \quad \text{if } x \notin Q_i^* \text{ and } l_i \geq C_5 \rho(x).$$

Proof. Take $\varphi \in \mathcal{D}_N^0(\mathbb{R}^n)$. Recall that η_i is supported in the cube \bar{Q}_i , and we have taken \bar{Q}_i to be strictly contained in Q_i^* . Thus if $x \notin Q_i^*$ and $\eta_i(y) \neq 0$, then there exists a positive constant C_4 such that $|x - y| \leq |x - x_i| \leq C_4 |x - y|$. On the other hand, the support property of φ requires that $\rho(x) > 2^{-l} \geq |x - y| \geq 2^{-11-n} l_i$. Hence, $|x - x_i| \leq C_4 2^{-l}$, $l_i < 2^{11+n} \rho(x) \equiv C_5 \rho(x)$ and $l_i < C_5 2^{-l}$. Pick some $w \in (2^{8+n} n Q_i) \cap \Omega^c$, and we discuss the following two cases.

Case I. If $L_3 \rho(x_i) \leq l_i < C_5 2^{-l} < C_5 \rho(x)$, then according to the Lemma 2.1 we have $l_i < C_5 C_0 (1 + C_4)^{k_0} \rho(x_i)$ and

$$\rho(\omega) \geq C_0^{-1} \left(1 + \frac{|\omega - x_i|}{\rho(x_i)}\right)^{-k_0} \rho(x_i) \geq C_0^{-1} (1 + 2^{8+n} n \sqrt{n} L_2)^{-k_0} \rho(x_i),$$

therefore, $l_i < a_1 \rho(w)$, where $a_1 > 1$ is a constant.

Now we define $\bar{l}_i = l_i / a_1 < \rho(w)$ and take $k_i \in \mathbb{Z}$ such that $2^{-k_i} \leq \bar{l}_i < 2^{-k_i+1}$, then for $\varphi \in \mathcal{D}_N^0(\mathbb{R}^n)$, $\phi(z) \equiv \varphi(2^{-k_i} z / 2^{-l})$ and $2^{-l} < \rho(x)$ we have

$$\begin{aligned} (b_i * \varphi_l)(x) &= 2^{ln} \int b_i(z) \varphi(2^l(x - z)) dz \\ &= 2^{ln} \int b_i(z) \phi(2^{k_i}(x - z)) dz \\ &= 2^{ln} \int b_i(z) \phi_{2^{k_i}(x-w)}(2^{k_i}(w - z)) dz \\ &= \frac{2^{ln}}{2^{k_i n}} (f * \Phi_{k_i})(w), \end{aligned}$$

where

$$\Phi(z) \equiv \phi_{2^{k_i}(x-w)}(z)\eta_i(w - 2^{-k_i}z), \quad \phi_{2^{k_i}(x-w)}(z) \equiv \phi(z + 2^{k_i}(x-w)).$$

Obviously, $\text{supp}\Phi \subset B(0, R_2)$, where $R_2 \equiv 2^{9+n}n^2a_1$. Notice that $l_i < C_52^{-l}$ and $|x - x_i| \leq C_42^{-l}$, we obtain

$$|(b_i * \varphi_l)(x)| \leq C \frac{2^{ln}}{2^{k_in}} \mathcal{M}_N f(w) \leq C \lambda \frac{2^{ln}}{2^{k_in}} \leq C \lambda \frac{l_i^{n+s+1}}{(l_i + |x - x_i|)^{n+s+1}}. \quad (4.7)$$

Case II. If $l_i < L_3\rho(x_i)$ and $\varphi \in \mathcal{D}_N^0(\mathbb{R}^n)$, taking $j_i \in \mathbb{Z}$ such that $2^{-j_i} \leq l_i < 2^{-j_i+1}$, then we define $\phi(z) = \varphi(2^{-j_i}z/2^{-l})$ and consider the Taylor expansion of ϕ of order s at the point $y = 2^{j_i}(x - w)$:

$$\phi(y + z) = \sum_{|\alpha| \leq s} \frac{\partial^\alpha \phi(y)}{\alpha!} z^\alpha + R_y(z),$$

where R_y denotes the remainder. Thus we get

$$\begin{aligned} (b_i * \varphi_l)(x) &= 2^{ln} \int b_i(z) \varphi(2^{ln}(x - z)) dz \\ &= 2^{ln} \int b_i(z) \phi(2^{j_in}(x - z)) dz \\ &= 2^{ln} \int b_i(z) R_{2^{j_i}(x-w)}(2^{j_i}(w - z)) dz \\ &= \frac{2^{ln}}{2^{j_in}} (f * \Phi_{j_i})(w) - 2^{ln} \int P_i(z) \eta_i(z) R_{2^{j_i}(x-w)}(2^{j_i}(w - z)) dz, \end{aligned} \quad (4.8)$$

where

$$\Phi(z) \equiv R_{2^{j_i}(x-w)}(z) \eta_i(w - 2^{-j_i}z).$$

Obviously, $\text{supp}\Phi \subset B_n \equiv B(0, R_2)$. Applying Lemma 4.4 to $\phi(z) = \varphi(2^{-j_i}z/2^{-l})$, $y = 2^{j_i}(x - w)$ and B_n , we have

$$\begin{aligned} \sup_{z \in B_n} \sup_{|\alpha| \leq N} |\partial^\alpha R_y(z)| &\leq C \sup_{z \in y + B_n} \sup_{s+1 \leq |\alpha| \leq N} |\partial^\alpha \phi(z)| \\ &\leq C \sup_{z \in y + B_n} \left(\frac{2^{-j_i}}{2^{-l}} \right)^{s+1} \sup_{s+1 \leq |\alpha| \leq N} |\partial^\alpha \varphi(2^{-j_i}z/2^{-l})| \\ &\leq C \left(\frac{2^{-j_i}}{2^{-l}} \right)^{s+1}. \end{aligned}$$

Notice that $l_i < C_52^{-l}$ and $|x - x_i| \leq C_42^{-l}$, therefore by (4.8), we obtain

$$\begin{aligned} (b_i * \varphi_l)(x) &\leq \frac{2^{ln}}{2^{j_in}} |(f * \Phi_{j_i})(w)| + 2^{ln} \int |P_i(z) \eta_i(z) R_{2^{j_i}(x-w)}(2^{j_i}(w - z))| dz \\ &\leq C \frac{2^{ln}}{2^{j_in}} \left(\mathcal{M}_N f(w) \|\Phi\|_{\mathcal{D}_N} + \lambda \sup_{z \in B_n} \sup_{|\alpha| \leq N} |\partial^\alpha R_y(z)| \right) \\ &\leq C \lambda \frac{l_i^{n+s+1}}{(l_i + |x - x_i|)^{n+s+1}}. \end{aligned} \quad (4.9)$$

By combining both cases, we obtain (4.6). \square

Lemma 4.6. *Let $\omega \in A_{\infty}^{\rho, \infty}(\mathbb{R}^n)$ and q_{ω} be as in (2.4). If $p \in (0, 1]$, $s \geq [n(q_{\omega}/p - 1)]$, $N > s$ and $N \geq N_{p, \omega}$, where $N_{p, \omega}$ is as in (3.29), then there exists a positive constant C_6 such that for all $f \in h_{\rho, N}^p(\omega)$, $\lambda > \inf_{x \in \mathbb{R}^n} \mathcal{M}_N f(x)$ and $i \in \mathbb{N}$,*

$$\int_{\mathbb{R}^n} (\mathcal{M}_N^0(b_i)(x))^p \omega(x) dx \leq C_6 \int_{Q_i^*} (\mathcal{M}_N(f)(x))^p \omega(x) dx. \quad (4.10)$$

Moreover the series $\sum_i b_i$ converges in $h_{\rho, N}^p(\omega)$ and

$$\int_{\mathbb{R}^n} \left(\mathcal{M}_N^0 \left(\sum_i b_i \right) (x) \right)^p \omega(x) dx \leq C_6 \int_{\Omega} (\mathcal{M}_N(f)(x))^p \omega(x) dx. \quad (4.11)$$

Proof. By the proof of Lemma 4.5, we know $|x - x_i| < C_4 \rho(x)$, $l_i < C_5 \rho(x)$ and $\rho(x) \leq C_0(1 + C_4)^{k_0} \rho(x_i)$, thus $Q_i^* \subset a_2 \rho(x_i) Q_i^0$, where $a_2 \equiv 2C_0(1 + C_4)^{k_0} \max\{C_4, C_5\}$ and $Q_i^0 \equiv Q(x_i, 1)$. Furthermore, we have

$$\begin{aligned} \int_{\mathbb{R}^n} (\mathcal{M}_N^0(b_i)(x))^p \omega(x) dx &\leq \int_{Q_i^*} (\mathcal{M}_N^0(b_i)(x))^p \omega(x) dx \\ &\quad + \int_{a_2 \rho(x_i) Q_i^0 \setminus Q_i^*} (\mathcal{M}_N^0(b_i)(x))^p \omega(x) dx. \end{aligned} \quad (4.12)$$

Notice that $s \geq [n(q_{\omega}/p - 1)]$ implies $2^{-n(q_{\omega} + \eta)} 2^{(s+n+1)p} > 1$ for sufficient small $\eta > 0$. By using Lemma 2.1 (iii) with $\omega \in A_{q_{\omega} + \eta}^{\rho, \infty}(\mathbb{R}^n)$, Lemma 4.5 and the fact that $\mathcal{M}_N(f)(x) > \lambda$ for all $x \in Q_i^*$, we have

$$\begin{aligned} \int_{a_2 \rho(x_i) Q_i^0 \setminus Q_i^*} (\mathcal{M}_N^0(b_i)(x))^p \omega(x) dx &\leq \sum_{k=0}^{k_0} \int_{2^k Q_i^* \setminus 2^{k-1} Q_i^*} (\mathcal{M}_N^0(b_i)(x))^p \omega(x) dx \\ &\leq \lambda^p \omega(Q_i^*) \sum_{k=0}^{k_0} [2^{-n(q_{\omega} + \eta) + (s+n+1)p}]^k \\ &\leq C \int_{Q_i^*} (\mathcal{M}_N f(x))^p \omega(x) dx, \end{aligned} \quad (4.13)$$

where $b = 1 + 2^{-(10+n)}$, $k_0 \in \mathbb{Z}$ such that $2^{k_0-1} b l_i \leq a_2 \rho(x_i) < 2^{k_0} b l_i$.

Combining the last two estimates we obtain (4.10), furthermore, we have

$$\sum_i \int_{\mathbb{R}^n} (\mathcal{M}_N^0(b_i)(x))^p \omega(x) dx \leq C \sum_i \int_{Q_i^*} (\mathcal{M}_N f(x))^p \omega(x) dx \leq C \int_{\Omega} (\mathcal{M}_N(f)(x))^p \omega(x) dx,$$

which together with complete of $h_{\rho, N}^p(\omega)$ (see Proposition 3.3) implies that $\sum_i b_i$ converges in $h_{\rho, N}^p(\omega)$. Therefore, the series $\sum_i b_i$ converges in $\mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{M}_N^0(\sum_i b_i)(x) \leq \sum_i \mathcal{M}_N^0(b_i)(x)$ by Proposition 3.2, which gives (4.11). This finishes the proof of Lemma 4.6. \square

Lemma 4.7. Let $\omega \in A_{\infty}^{\rho, \infty}(\mathbb{R}^n)$ and q_ω be as in (2.4), $s \in \mathbb{Z}_+$, and integer $N \geq 2$. If $q \in (q_\omega, \infty]$ and $f \in L_\omega^q(\mathbb{R}^n)$, then the series $\sum_i b_i$ converges in $L_\omega^q(\mathbb{R}^n)$ and there exists a positive constant C_7 , independent of f and λ , such that

$$\left\| \sum_i |b_i| \right\|_{L_\omega^q(\mathbb{R}^n)} \leq C_7 \|f\|_{L_\omega^q(\mathbb{R}^n)}.$$

Proof. The proof for $q = \infty$ is similar to that for $q \in (q_\omega, \infty)$. So we only give the proof for $q \in (q_\omega, \infty)$. Set $F_1 = \{i \in \mathbb{N} : l_i \geq L_3 \rho(x_i)\}$ and $F_2 = \{i \in \mathbb{N} : l_i < L_3 \rho(x_i)\}$. By Lemma 4.2, for $i \in F_2$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} |b_i(x)|^q \omega(x) dx &\leq \int_{Q_i^*} |f(x)|^q \omega(x) dx + \int_{Q_i^*} |P_i(x) \eta_i(x)|^q \omega(x) dx \\ &\leq \int_{Q_i^*} |f(x)|^q \omega(x) dx + \lambda^q \omega(Q_i^*). \end{aligned}$$

For $i \in F_1$, we have

$$\int_{\mathbb{R}^n} |b_i(x)|^q \omega(x) dx \leq \int_{Q_i^*} |f(x)|^q \omega(x) dx.$$

From these, we obtain

$$\begin{aligned} \sum_i \int_{\mathbb{R}^n} |b_i(x)|^q \omega(x) dx &= \sum_{i \in F_1} \int_{\mathbb{R}^n} |b_i(x)|^q \omega(x) dx + \sum_{i \in F_2} \int_{\mathbb{R}^n} |b_i(x)|^q \omega(x) dx \\ &\leq \sum_i \int_{Q_i^*} |f(x)|^q \omega(x) dx + C \sum_{i \in F_2} \lambda^q \omega(Q_i^*) \\ &\leq \sum_i \int_{Q_i^*} |f(x)|^q \omega(x) dx + C \lambda^q \omega(\Omega) \\ &\leq C \int_{\mathbb{R}^n} |f(x)|^q \omega(x) dx. \end{aligned}$$

Combining above estimates with the fact that $\{b_i\}_i$ have finite covering, we obtain

$$\left\| \sum_i |b_i| \right\|_{L_\omega^q(\mathbb{R}^n)} \leq C_7 \|f\|_{L_\omega^q(\mathbb{R}^n)}.$$

This finishes the proof. \square

Lemma 4.8. If $N > s \geq 0$ and $\sum_i b_i$ converges in $\mathcal{D}'(\mathbb{R}^n)$, then there exists a positive constant C_8 , independent of f and λ , such that for all $x \in \mathbb{R}^n$,

$$\mathcal{M}_N^0(g)(x) \leq \mathcal{M}_N^0(f)(x) \chi_{\Omega^c}(x) + C_8 \lambda \sum_i \frac{l_i^{n+s+1}}{(l_i + |x - x_i|)^{n+s+1}} \chi_{\{|x - x_i| < C_4 \rho(x)\}}(x) + C_8 \lambda \chi_\Omega(x),$$

where x_i is the center of Q_i and C_4 is as in Lemma 4.5.

Proof. If $x \notin \Omega$, since

$$\mathcal{M}_N^0(g)(x) \leq \mathcal{M}_N^0(f)(x) + \sum_i \mathcal{M}_N^0(b_i)(x),$$

by Lemma 4.5, we obtain

$$\mathcal{M}_N^0(g)(x) \leq \mathcal{M}_N^0(f)(x)\chi_{\Omega^c}(x) + C\lambda \sum_i \frac{l_i^{n+s+1}}{(l_i + |x - x_i|)^{n+s+1}} \chi_{\{|x-x_i| < C_4\rho(x)\}}(x).$$

If $x \in \Omega$, take $k \in \mathbb{N}$ such that $x \in Q_k^*$. Let $J \equiv \{i \in \mathbb{N} : Q_i^* \cap Q_k^* \neq \emptyset\}$. Then the cardinality of J is bounded by L . By Lemma 4.5, we have

$$\sum_{i \notin J} \mathcal{M}_N^0(b_i)(x) \leq C\lambda \sum_{i \notin J} \frac{l_i^{n+s+1}}{(l_i + |x - x_i|)^{n+s+1}} \chi_{\{|x-x_i| < C_4\rho(x)\}}(x).$$

It suffices to estimate the grand maximal function of $g + \sum_{i \notin J} b_i = f - \sum_{i \in J} b_i$. Take $\varphi \in \mathcal{D}_N^0(\mathbb{R}^n)$ and $l \in \mathbb{Z}$ such that $0 < 2^{-l} < \rho(x)$, then we write

$$\begin{aligned} \left(f - \sum_{i \in J} b_i\right) * \varphi_l(x) &= (f\xi) * \varphi_l(x) + \left(\sum_{i \in J} P_i \eta_i\right) * \varphi_l(x) \\ &= f * \tilde{\Phi}_l(w) + \left(\sum_{i \in J} P_i \eta_i\right) * \varphi_l(x), \end{aligned} \tag{4.14}$$

where $w \in (2^{8+n}Q_k) \cap \Omega^c$, $\xi = 1 - \sum_{i \in J} \eta_i$ and

$$\tilde{\Phi}(z) \equiv \varphi(z + 2^l(x - w))\xi(w - 2^{-l}z).$$

Since for $N \geq 2$ there is a constant $C > 0$ such that $\|\varphi\|_{L^1(\mathbb{R}^n)} \leq C$ for all $\varphi \in \mathcal{D}_N^0(\mathbb{R}^n)$ and Lemma 4.1, we have

$$\left|\left(\sum_{i \in J} P_i \eta_i\right) * \varphi_l(x)\right| \leq C\lambda.$$

Finally, we estimate $f * \Phi_l(w)$. There are two cases: If $2^{-l} \leq 2^{-(11+n)}l_k$, then $f * \Phi_l(w) = 0$, because ξ vanishes in Q_k^* and φ_l is supported in $B(0, 2^{-l})$. On the other hand, if $2^{-l} \geq 2^{-(11+n)}l_k$, then there exists a positive constant $a_3 > 1$ such that $2^{-l} < \rho(x) < a_3\rho(w)$. Take $\Phi(x) \equiv \tilde{\Phi}(x/2^{m_1})$ and $m_1 \in \mathbb{N}$ satisfying $2^{m_1-1} \leq a_3 < 2^{m_1}$, then $\text{supp}\Phi \subset B(0, R_3)$ where $R_3 \equiv 2^{3(11+n)}a_3$, and $\|\Phi\|_{\mathcal{D}_N} \leq C$. Therefore, $2^{-l-m_1} < \rho(x)/a_3 < \rho(w)$ and

$$\left|(f * \tilde{\Phi}_l)(w)\right| = 2^{-m_1 n} |(f * \Phi_{l+m_1})(w)| \leq C\mathcal{M}_N f(w)\|\Phi\|_{\mathcal{D}_N} \leq C\lambda.$$

According to above estimates, we have

$$\left|(f - \sum_{i \in J} b_i) * \varphi_l\right| \leq C\lambda,$$

then we can get

$$\mathcal{M}_N^0\left((f - \sum_{i \in J} b_i)\right)(x) \leq C\lambda.$$

This finishes the proof of the lemma. \square

Lemma 4.9. *Let $\omega \in A_{\infty}^{\rho, \infty}(\mathbb{R}^n)$, q_{ω} be as in (2.4), $q \in (q_{\omega}, \infty)$, $p \in (0, 1]$ and $N \geq N_{p, \omega}$, where $N_{p, \omega}$ is as in (3.29).*

(i) *If $N > s \geq [n(q_{\omega}/p - 1)]$ and $f \in h_{\rho, N}^p(\omega)$, then $\mathcal{M}_N^0(g) \in L_{\omega}^q(\mathbb{R}^n)$ and there exists a positive constant C_9 , independent of f and λ , such that*

$$\int_{\mathbb{R}^n} [\mathcal{M}_N^0(g)(x)]^q \omega(x) dx \leq C_9 \lambda^{q-p} \int_{\mathbb{R}^n} [\mathcal{M}_N(f)(x)]^p \omega(x) dx.$$

(ii) *If $N \geq 2$ and $f \in L_{\omega}^q(\mathbb{R}^n)$, then $g \in L_{\omega}^{\infty}(\mathbb{R}^n)$ and there exists a positive constant C_{10} , independent of f and λ , such that $\|g\|_{L_{\omega}^{\infty}} \leq C_{10} \lambda$.*

Proof. We first prove (i). Since $f \in h_{\rho, N}^p(\omega)$, by Lemma 4.6 and Proposition 3.2, $\sum_i b_i$ converges in both $h_{\rho, N}^p(\omega)$ and $\mathcal{D}'(\mathbb{R}^n)$. Notice that $s \geq [n(q_{\omega}/p - 1)]$, by Lemma 4.8 and the proof of Lemma 4.6, we get

$$\begin{aligned} \int_{\mathbb{R}^n} (\mathcal{M}_N^0(g)(x))^q \omega(x) dx &\leq C \lambda^q \sum_i \int_{\mathbb{R}^n} \left[\frac{l_i^{(n+s+1)}}{(l_i + |x - x_i|)^{(n+s+1)}} \chi_{B(x_i, a_2 \rho(x_i))}(x) \right]^q \omega(x) dx \\ &\quad + C \lambda^q \int_{\mathbb{R}^n} \chi_{\Omega}(x) \omega(x) dx + \int_{\Omega^c} (\mathcal{M}_N(f)(x))^q \omega(x) dx \\ &\leq C \lambda^q \sum_i \omega(Q_i^*) + C \lambda^q \omega(\Omega) + \int_{\Omega^c} (\mathcal{M}_N(f)(x))^q \omega(x) dx \\ &\leq C \lambda^q \omega(\Omega) + C \lambda^{q-p} \int_{\Omega^c} (\mathcal{M}_N(f)(x))^p \omega(x) dx \\ &\leq C_9 \lambda^{q-p} \int_{\mathbb{R}^n} (\mathcal{M}_N(f)(x))^p \omega(x) dx. \end{aligned}$$

Thus, (i) holds.

Next, we prove (ii). If $f \in L_{\omega}^q(\mathbb{R}^n)$, then g and $\{b_i\}_i$ are functions. By Lemma 4.7, we know that $\sum_i b_i$ converges in $L_{\omega}^q(\mathbb{R}^n)$ and hence in $\mathcal{D}'(\mathbb{R}^n)$ by Lemma 2.5(ii). Write

$$g = f - \sum_i b_i = f \left(1 - \sum_i \eta_i \right) + \sum_{i \in F_2} P_i \eta_i = f \chi_{\Omega^c} + \sum_{i \in F_2} P_i \eta_i.$$

By Lemma 4.3, we have $|g(x)| \leq C \lambda$ for all $x \in \Omega$, and by Proposition 3.1(i), we also have $|g(x)| = |f(x)| \leq \mathcal{M}_N f(x) \leq \lambda$ for almost everywhere $x \in \Omega^c$. Therefore, $\|g\|_{L_{\omega}^{\infty}(\mathbb{R}^n)} \leq C_{10} \lambda$ which yields (ii). \square

Corollary 4.1. *Let $\omega \in A_{\infty}^{\rho, \infty}(\mathbb{R}^n)$ and q_{ω} be as in (2.4). If $q \in (q_{\omega}, \infty)$, $p \in (0, 1]$ and $N \geq N_{p, \omega}$, where $N_{p, \omega}$ is as in (3.29), then $h_{\rho, N}^p(\omega) \cap L_{\omega}^q(\mathbb{R}^n)$ is dense in $h_{\rho, N}^p(\omega)$.*

Proof. Let $f \in h_{\rho, N}^p(\omega)$. For any $\lambda > \inf_{x \in \mathbb{R}^n} \mathcal{M}_N f(x)$, let $f = g^{\lambda} + \sum_i b_i^{\lambda}$ be the Calderón-Zygmund decomposition of f of degree s with $[n(q_{\omega}/p - 1)] \leq s < N$ and height λ associated

to $\mathcal{M}_N f$. By Lemma 4.6, we have

$$\left\| \sum_i b_i^\lambda \right\|_{h_{\rho,N}^p(\omega)} \leq C \int_{\{x \in \mathbb{R}^n : \mathcal{M}_N f(x) > \lambda\}} (\mathcal{M}_N f(x))^p \omega(x) dx.$$

Therefore, $g^\lambda \rightarrow f$ in $h_{\rho,N}^p(\omega)$ as $\lambda \rightarrow \infty$. Moreover, by Lemma 4.9, we have $\mathcal{M}_N^0(g^\lambda) \in L_\omega^q(\mathbb{R}^n)$, which combined with Proposition 3.1(ii) implies $g^\lambda \in L_\omega^q(\mathbb{R}^n)$. Thus, Corollary 4.1 is proved. \square

5 Weighted atomic decompositions of $h_{\rho,N}^p(\omega)$

In this section, we establish the equivalence between $h_{\rho,N}^p(\omega)$ and $h_{\rho}^{p,q,s}(\omega)$ by using the Calderón-Zygmund decomposition associated to the local grand maximal function stated in Section 4.

Let $\omega \in A_\infty^{\rho,\infty}(\mathbb{R}^n)$, q_ω be as in (2.4), $p \in (0, 1]$, $N \geq N_{p,\omega}$, $s \equiv [n(q_\omega/p - 1)]$ and $f \in h_{\rho,N}^p(\omega)$. Take $m_0 \in \mathbb{Z}$ such that $2^{m_0-1} \leq \inf_{x \in \mathbb{R}^n} \mathcal{M}_N f(x) < 2^{m_0}$, if $\inf_{x \in \mathbb{R}^n} \mathcal{M}_N f(x) = 0$, write $m_0 = -\infty$. For each integer $m \geq m_0$ consider the Calderón-Zygmund decomposition of f of degree s and height $\lambda = 2^m$ associated to $\mathcal{M}_N f$, namely

$$f = g^m + \sum_{i \in \mathbb{N}} b_i^m, \quad (5.1)$$

and

$$\Omega_m \equiv \{x \in \mathbb{R}^n : \mathcal{M}_N f(x) > 2^m\}, \quad Q_i^m \equiv Q_i^m.$$

In this section, we write $\{Q_i\}_i$, $\{\eta_i\}_i$, $\{P_i\}_i$ and $\{b_i\}_i$, respectively, as $\{Q_i^m\}_i$, $\{\eta_i^m\}_i$, $\{P_i^m\}_i$ and $\{b_i^m\}_i$. The center and the sidelength of Q_i^m are respectively denoted by x_i^m and l_i^m . Recall that for all i and m ,

$$\sum_i \eta_i^m = \chi_{\Omega_m}, \quad \text{supp}(b_i^m) \subset \text{supp}(\eta_i^m) \subset Q_i^{m*}, \quad (5.2)$$

$\{Q_i^{m*}\}_i$ has the bounded interior property, and for all $P \in \mathcal{P}_s(\mathbb{R}^n)$,

$$\langle f, P\eta_i^m \rangle = \langle P_i^m, P\eta_i^m \rangle. \quad (5.3)$$

For each integer $m \geq m_0$ and $i, j \in \mathbb{N}$, let $P_{i,j}^{m+1}$ be the orthogonal projection of $(f - P_j^{m+1})\eta_i^m$ on $\mathcal{P}_s(\mathbb{R}^n)$ with respect to the norm

$$\|P\|_j^2 \equiv \frac{1}{\int_{\mathbb{R}^n} \eta_j^{m+1}(y) dy} \int_{\mathbb{R}^n} |P(x)|^2 \eta_j^{m+1}(x) dx,$$

namely, $P_{i,j}^{m+1}$ is the unique polynomial of $\mathcal{P}_s(\mathbb{R}^n)$ such that for any $P \in \mathcal{P}_s(\mathbb{R}^n)$,

$$\langle (f - P_j^{m+1})\eta_i^k, P\eta_j^{m+1} \rangle = \int_{\mathbb{R}^n} P_{i,j}^{m+1}(x) P(x) \eta_j^{m+1}(x) dx. \quad (5.4)$$

In what follows, we denote $Q_i^{m*} = (1 + 2^{-(10+n)})Q_i^m$,

$$\begin{aligned} E_1^m &\equiv \{i \in \mathbb{N} : l_i^m \geq \rho(x_i^m)/(2^5 n)\}, & E_2^k &\equiv \{i \in \mathbb{N} : l_i^m < \rho(x_i^m)/(2^5 n)\}, \\ F_1^k &\equiv \{i \in \mathbb{N} : l_i^m \geq L_3 \rho(x_i^m)\}, & F_2^k &\equiv \{i \in \mathbb{N} : l_i^m < L_3 \rho(x_i^m)\}, \end{aligned}$$

where $L_3 = 2^{k_0} C_0$ is as in Section 4.

Observe that

$$P_{i,j}^{m+1} \neq 0 \quad \text{if and only if} \quad Q_i^{m*} \cap Q_j^{(m+1)*} \neq \emptyset. \quad (5.5)$$

Indeed, this follows directly from the definition of $P_{i,j}^{m+1}$. The following Lemmas 5.1-5.3 can be proved by similar methods of Lemmas 5.1-5.3 in [28].

Lemma 5.1. *Notice that $\Omega_{m+1} \subset \Omega_m$, then*

- (i) *If $Q_i^{m*} \cap Q_j^{(m+1)*} \neq \emptyset$, then $l_j^{m+1} \leq 2^4 \sqrt{n} l_i^m$ and $Q_j^{(m+1)*} \subset 2^6 n Q_i^{k*} \subset \Omega_m$.*
- (ii) *There exists a positive integer L such that for each $i \in \mathbb{N}$, the cardinality of $\{j \in \mathbb{N} : Q_i^{m*} \cap Q_j^{(m+1)*} \neq \emptyset\}$ is bounded by L .*

Lemma 5.2. *There exists a positive constant C such that for all $i, j \in \mathbb{N}$ and integer $m \geq m_0$ with $l_j^{m+1} < L_3 \rho(x_j^{m+1})$,*

$$\sup_{y \in \mathbb{R}^n} |P_{i,j}^{m+1}(y) \eta_j^{m+1}(y)| \leq C 2^{m+1}. \quad (5.6)$$

Lemma 5.3. *For any $k \in \mathbb{Z}$ with $m \geq m_0$,*

$$\sum_{i \in \mathbb{N}} \left(\sum_{j \in F_2^{m+1}} P_{i,j}^{m+1} \eta_j^{m+1} \right) = 0,$$

where the series converges both in $\mathcal{D}'(\mathbb{R}^n)$ and pointwise.

The following lemma gives the weighted atomic decomposition for a dense subspace of $h_{\rho,N}^p(\omega)$.

Lemma 5.4. *Let $\omega \in A_{\infty}^{\rho,\infty}(\mathbb{R}^n)$, q_ω and $N_{p,\omega}$ be respectively as in (2.4) and (3.29). If $p \in (0, 1]$, $s \geq [n(q_\omega/p - 1)]$, $N > s$ and $N \geq N_{p,\omega}$, then for any $f \in (L_\omega^q(\mathbb{R}^n) \cap h_{\rho,N}^p(\omega))$, there exists numbers $\lambda_0 \in \mathbb{C}$ and $\{\lambda_i^m\}_{m \geq k_0, i} \subset \mathbb{C}$, $(p, \infty, s)_\omega$ -atoms $\{a_i^m\}_{m \geq m_0, i}$ and a single atom a_0 such that*

$$f = \sum_{m \geq m_0} \sum_i \lambda_i^m a_i^m + \lambda_0 a_0, \quad (5.7)$$

where the series converges almost everywhere and in $\mathcal{D}'(\mathbb{R}^n)$. Moreover, there exists a positive constant C , independent of f , such that

$$\sum_{m \geq m_0, i} |\lambda_i^m|^p + |\lambda_0|^p \leq C \|f\|_{h_{\rho,N}^p(\omega)}^p. \quad (5.8)$$

Proof. Let $f \in (L_\omega^q(\mathbb{R}^n) \cap h_{\rho, N}^p(\omega))$. We first consider the case $m_0 = -\infty$. As above, for each $m \in \mathbb{Z}$, f has a Calderón-Zygmund decomposition of degree s and height $\lambda = 2^m$ associated to $\mathcal{M}_N(f)$ as in (5.1), namely, $f = g^m + \sum_i b_i^m$. By Corollary 4.1 and Proposition 3.1, $g^m \rightarrow f$ in both $h_{\rho, N}^p(\omega)$ and $\mathcal{D}'(\mathbb{R}^n)$ as $m \rightarrow \infty$. By Lemma 4.9 (i), $\|g^m\|_{L_\omega^q(\mathbb{R}^n)} \rightarrow 0$ as $m \rightarrow -\infty$, and moreover, by Lemma 2.5 (ii), $g^m \rightarrow 0$ in $\mathcal{D}'(\mathbb{R}^n)$ as $m \rightarrow -\infty$. Therefore,

$$f = \sum_{m=-\infty}^{\infty} (g^{m+1} - g^m) \quad (5.9)$$

in $\mathcal{D}'(\mathbb{R}^n)$. Moreover, since $\text{supp}(\sum_i b_i^m) \subset \Omega_m$ and $\omega(\Omega_m) \rightarrow 0$ as $m \rightarrow \infty$, then $g^m \rightarrow f$ almost everywhere as $m \rightarrow \infty$. Thus, (5.9) also holds almost everywhere. By Lemma 5.3 and $\sum_i \eta_i^m b_j^{m+1} = \chi_{\Omega_m} b_j^{m+1} = b_j^{m+1}$ for all j , then we have

$$\begin{aligned} g^{m+1} - g^m &= \left(f - \sum_j b_j^{m+1}\right) - \left(f - \sum_i b_i^m\right) \\ &= \sum_i b_i^m - \sum_j b_j^{m+1} + \sum_i \left(\sum_{j \in F_2^{m+1}} P_{i,j}^{m+1} \eta_j^{m+1} \right) \\ &= \sum_i \left[b_i^m - \sum_j b_j^{m+1} \eta_i^m + \sum_{j \in F_2^{m+1}} P_{i,j}^{m+1} \eta_j^{m+1} \right] \equiv \sum_i h_i^m, \end{aligned} \quad (5.10)$$

where all the series converge in both $\mathcal{D}'(\mathbb{R}^n)$ and almost everywhere. Furthermore, from the definitions of b_j^m and b_j^{m+1} , we infer that when $l_i^m < L_3 \rho(x_i^m)$,

$$h_i^m = f \chi_{\Omega_{m+1}^c} \eta_i^m - P_i^m \eta_i^m + \sum_{j \in F_2^{m+1}} P_j^{m+1} \eta_i^m \eta_j^{m+1} + \sum_{j \in F_2^{m+1}} P_{i,j}^{m+1} \eta_j^{m+1}, \quad (5.11)$$

and when $l_i^m \geq L_3 \rho(x_i^m)$,

$$h_i^m = f \chi_{\Omega_{m+1}^c} \eta_i^m + \sum_{j \in F_2^{m+1}} P_j^{m+1} \eta_i^m \eta_j^{m+1} + \sum_{j \in F_2^{m+1}} P_{i,j}^{m+1} \eta_j^{m+1}. \quad (5.12)$$

By Proposition 3.1(i), we know that for almost every $x \in \Omega_{m+1}^c$,

$$|f(x)| \leq \mathcal{M}_N(f)(x) \leq 2^{m+1},$$

which combined with Lemma 4.2, Lemma 5.1(ii), Lemma 5.2, (5.11) and (5.12) implies that there exists a positive constant C_{11} such that for all $i \in \mathbb{N}$,

$$\|h_i^m\|_{L_\omega^\infty(\mathbb{R}^n)} \leq C_{11} 2^m. \quad (5.13)$$

Next, we show that for each i and m , h_i^m is either a multiple of a $(p, \infty, s)_\omega$ -atom or a finite linear combination of $(p, \infty, s)_\omega$ -atom by considering the following two cases for i .

Case I. For $i \in E_1^m$, $l_i^m \geq \rho(x_i^m)/2^5 n$. Clearly, h_i^m is supported in a cube \tilde{Q}_i^m that contains Q_i^{m*} as well as all the $Q_j^{(m+1)*}$ that intersect Q_i^{m*} . In fact, observe that if $Q_i^{m*} \cap Q_j^{(m+1)*} \neq \emptyset$

\emptyset , by Lemma 5.1, we have $Q_j^{(m+1)*} \subset 2^6 n Q_i^{m*} \subset \Omega_m$, thus, we set $\tilde{Q}_i^m \equiv 2^6 n Q_i^{m*}$. Since $l(\tilde{Q}_i^m) \geq 2\rho(x_i^m)$, by the same method of Lemma 3.1 in [30], \tilde{Q}_i^m can be decomposed into finite disjoint cubes $\{Q_{i,k}^m\}_k$ such that $\tilde{Q}_i^m = \bigcup_{k=1}^{n_i} Q_{i,k}^m$ and $l_{i,k}^m/4 < \rho(x) \leq C_0(3\sqrt{n})^{k_0} l_{i,k}^m$ for some $x \in Q_{i,k}^m = Q(x_{i,k}^m, l_{i,k}^m)$, where C_0, k_0 are constants given in Lemma 2.1. Moreover, by Lemma 2.1, we also have $l_{i,k}^m \leq L_1 \rho(x_{i,k}^m)$ and $l_{i,k}^m > L_2 \rho(x_{i,k}^m)$. Therefore, let

$$\lambda_{i,k}^m \equiv C_{11} 2^m [\omega(Q_{i,k}^m)]^{1/p} \quad \text{and} \quad a_{i,k}^m \equiv (\lambda_{i,k}^m)^{-1} \frac{h_i^m \chi_{Q_{i,k}^m}}{\sum_{k=1}^{n_i} \chi_{Q_{i,k}^m}},$$

then $\text{supp } a_{i,k}^m \subset Q_{i,k}^m$ and $\|a_{i,k}^m\|_{L^\infty(\mathbb{R}^n)} \leq [\omega(Q_{i,k}^m)]^{-1/p}$, hence each $a_{i,k}^m$ is a $(p, \infty, s)_\omega$ -atom and $h_i^m = \sum_{k=1}^{n_i} \lambda_{i,k}^m a_{i,k}^m$.

Case II. For $i \in E_2^m$, if $j \in F_1^{m+1}$, we claim that $Q_i^{m*} \cap Q_j^{(m+1)*} = \emptyset$. In fact, if $Q_i^{m*} \cap Q_j^{(m+1)*} \neq \emptyset$, by Lemma 5.1 (i), we know $l_j^{m+1} \leq 2^4 \sqrt{n} l_i^m$ then we can deduce that $l_i^m < l_i^m / 2\sqrt{n}$ which is a contradiction, hence the claim is true. Thus, we have

$$\begin{aligned} h_i^m &= (f - P_i^m) \eta_i^m - \sum_{j \in F_1^{m+1}} f \eta_j^{m+1} \eta_i^m - \sum_{j \in F_2^{m+1}} (f - P_j^{m+1}) \eta_j^{m+1} \eta_i^m \\ &\quad + \sum_{j \in F_2^{m+1}} P_{i,j}^{m+1} \eta_j^{m+1} \\ &= (f - P_i^m) \eta_i^m - \sum_{j \in F_2^{m+1}} \left\{ (f - P_j^{m+1}) \eta_j^{m+1} \eta_i^m - P_{i,j}^{m+1} \eta_j^{m+1} \right\}. \end{aligned} \quad (5.14)$$

Let $\tilde{Q}_i^m \equiv 2^6 n Q_i^{m*}$, then $l(\tilde{Q}_i^m) < L_1 \rho(x_i^m)$ and $\text{supp } h_i^m \subset \tilde{Q}_i^m$. Moreover, h_i^m satisfies the desired moment conditions, which are deduced from the moment conditions of $(f - P_i^m) \eta_i^m$ (see (5.3)) and $(f - P_j^{m+1}) \eta_j^{m+1} \eta_i^m - P_{i,j}^{m+1} \eta_j^{m+1}$ (see (5.4)). Let $\lambda_i^m \equiv C_{11} 2^m [\omega(\tilde{Q}_i^m)]^{1/p}$ and $a_i^m \equiv (\lambda_i^m)^{-1} h_i^m$, then a_i^m is a $(p, \infty, s)_\omega$ -atom.

Thus, from (5.9), (5.10), Case I and Case II, we infer that

$$f = \sum_{m \in \mathbb{Z}} \left(\sum_{i \in E_1^m} \left(\sum_{k=1}^{n_i} \lambda_{i,k}^m a_{i,k}^m \right) + \sum_{i \in E_2^m} \lambda_i^m a_i^m \right)$$

holds in both $\mathcal{D}'(\mathbb{R}^n)$ and almost everywhere. Moreover, by Lemma 2.4, we get

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \left[\sum_{i \in E_1^m} \left[\sum_{k=1}^{n_i} |\lambda_{i,k}^m|^p \right] + \sum_{i \in E_2^m} |\lambda_i^m|^p \right] &\leq C \sum_{k \in \mathbb{Z}} 2^{mp} \left[\sum_{i \in E_1^m} \left[\sum_{k=1}^{n_i} \omega(Q_{i,k}^m) \right] + \sum_{i \in E_2^m} \omega(\tilde{Q}_i^m) \right] \\ &\leq C \sum_{k \in \mathbb{Z}} 2^{mp} \left[\sum_{i \in E_1^m} \omega(\tilde{Q}_i^m) + \sum_{i \in E_2^m} \omega(\tilde{Q}_i^m) \right] \\ &\leq C \sum_{m \in \mathbb{Z}} \sum_{i \in \mathbb{N}} 2^{mp} \omega(\tilde{Q}_i^m) \end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{m \in \mathbb{Z}} \sum_{i \in \mathbb{N}} 2^{mp} \omega(Q_i^{m*}) \\
&\leq C \sum_{m \in \mathbb{Z}} 2^{mp} \omega(\Omega_m) \\
&\leq C \|\mathcal{M}_N(f)\|_{L_\omega^p(\mathbb{R}^n)}^p = C \|f\|_{h_{\rho,N}^p(\omega)}^p,
\end{aligned}$$

which implies (5.8) in the case that $m_0 = -\infty$.

Finally, we consider the case that $k_0 > -\infty$. In this case, by $f \in h_{\rho,N}^p(\omega)$, we see that $\omega(\mathbb{R}^n) < \infty$. Adapting the previous arguments, we conclude that

$$f = \sum_{m=m_0}^{\infty} (g^{m+1} - g^m) + g^{m_0} \equiv \tilde{f} + g^{m_0}. \quad (5.15)$$

For the function \tilde{f} , we have the same $(p, \infty, s)_\omega$ atomic decomposition as above

$$\tilde{f} = \sum_{m \geq m_0, i} \lambda_i^m a_i^m, \quad (5.16)$$

and

$$\sum_{m \geq m_0} \sum_{i \in \mathbb{N}} |\lambda_i^m|^p \leq C \|f\|_{h_{\rho,N}^p(\omega)}^p. \quad (5.17)$$

For the function g^{m_0} , by Lemma 4.9 (ii), we have

$$\|g^{m_0}\|_{L_\omega^\infty(\mathbb{R}^n)} \leq C_{10} 2^{m_0} \leq 2C_{10} \inf_{x \in \mathbb{R}^n} \mathcal{M}_N f(x), \quad (5.18)$$

where C_{10} is the same as in Lemma 4.9 (ii). Let $\lambda_0 \equiv C_{10} 2^{m_0} [\omega(\mathbb{R}^n)]^{1/p}$ and $a_0 \equiv \lambda_0^{-1} g^{m_0}$, then

$$\|a_0\|_{L_\omega^\infty(\mathbb{R}^n)} \leq [\omega(\mathbb{R}^n)]^{-1/p} \quad \text{and} \quad |\lambda_0|^p \leq (2C_{10})^p \|f\|_{h_{\rho,N}^p(\omega)}^p. \quad (5.19)$$

Thus, a_0 is a $(p, \infty)_\omega$ -single-atom and $g^{m_0} = \lambda_0 a_0$, which together with (5.15) and (5.16) implies (5.7) in the case that $k_0 > -\infty$. Furthermore, by combining (5.17) with (5.19), we obtain

$$\sum_{m \geq m_0} \sum_{i \in \mathbb{N}} |\lambda_i^m|^p + |\lambda_0|^p \leq C \|f\|_{h_{\rho,N}^p(\omega)}^p.$$

The proof of the lemma is complete. \square

Now we state the weighted atomic decompositions of $h_{\rho,N}^p(\omega)$.

Theorem 5.1. *Let $\omega \in A_\infty^{\rho,\infty}(\mathbb{R}^n)$, q_ω and $N_{p,\omega}$ be respectively as in (2.4) and (3.29). If $q \in (q_\omega, \infty]$, $p \in (0, 1]$, and integers s and N satisfy $N \geq N_{p,\omega}$ and $N > s \geq [n(q_\omega/p - 1)]$, then $h_\rho^{p,q,s}(\omega) = h_{\rho,N}^p(\omega) = h_{\rho,N_{p,\omega}}^p(\omega)$ with equivalent norms.*

Proof. It is easy to see that

$$h_{\rho}^{p,\infty,\bar{s}}(\omega) \subset h_{\rho}^{p,q,s}(\omega) \subset h_{\rho,N_p,\omega}^p(\omega) \subset h_{\rho,N}^p(\omega) \subset h_{\rho,\bar{N}}^p(\omega),$$

where \bar{s} is an integer no less than s and \bar{N} is an integer larger than N , and the inclusions are continuous. Thus, to prove Theorem 5.1, it suffices to prove that for any $N > s \geq [n(q_{\omega}/p-1)]$, $h_{\rho,N}^p(\omega) \subset h_{\rho}^{p,\infty,s}(\omega)$, and for all $f \in h_{\rho,N}^p(\omega)$, $\|f\|_{h_{\rho}^{p,\infty,s}(\omega)} \leq C\|f\|_{h_{\rho,N}^p(\omega)}$.

Let $f \in h_{\rho,N}^p(\omega)$. By Corollary 4.1, there exists a sequence of functions $\{f_m\}_{m \in \mathbb{N}} \subset (h_{\rho,N}^p(\omega) \cap L_{\omega}^q(\mathbb{R}^n))$ such that for all $m \in \mathbb{N}$,

$$\|f_m\|_{h_{\rho,N}^p(\omega)} \leq 2^{-m}\|f\|_{h_{\rho,N}^p(\omega)} \quad (5.20)$$

and $f = \sum_{m \in \mathbb{N}} f_m$ in $h_{\rho,N}^p(\omega)$. By Lemma 5.4, for each $m \in \mathbb{N}$, f_m has an atomic decomposition

$$f_m = \sum_{i \in \mathbb{Z}_+} \lambda_i^m a_i^m$$

in $\mathcal{D}'(\mathbb{R}^n)$ with

$$\sum_{i \in \mathbb{Z}_+} |\lambda_i^m|^p \leq C\|f_m\|_{h_{\rho,N}^p(\omega)}^p,$$

where $\{\lambda_i^m\}_{i \in \mathbb{Z}_+} \subset \mathbb{C}$, $\{a_i^m\}_{i \in \mathbb{N}}$ are $(p, \infty, s)_{\omega}$ -atoms and a_0^m is a $(p, \infty)_{\omega}$ -single-atom.

Let

$$\tilde{\lambda}_0 \equiv [\omega(\mathbb{R}^n)]^{1/p} \sum_{m=1}^{\infty} |\lambda_0^m| \|a_0^m\|_{L_{\omega}^{\infty}(\mathbb{R}^n)} \quad \text{and} \quad \tilde{a}_0 \equiv (\tilde{\lambda}_0)^{-1} \sum_{m=1}^{\infty} \lambda_0^m a_0^m.$$

Then

$$\tilde{\lambda}_0 \tilde{a}_0 = \sum_{m=1}^{\infty} \lambda_0^m a_0^m.$$

It is easy to see that

$$\|\tilde{a}_0\|_{L_{\omega}^{\infty}(\mathbb{R}^n)} \leq [\omega(\mathbb{R}^n)]^{-1/p},$$

which implies that \tilde{a}_0 is a $(\rho, \infty)_{\omega}$ -single-atom. Since $\|a_0^m\|_{L_{\omega}^{\infty}(\mathbb{R}^n)} \leq (\omega(\mathbb{R}^n))^{-1/p}$ and

$$|\lambda_0^m| \leq C\|f_m\|_{h_{\rho,N}^p(\omega)} \leq C2^{-m}\|f\|_{h_{\rho,N}^p(\omega)},$$

we have

$$|\tilde{\lambda}_0| \leq C \left(\sum_{m=1}^{\infty} 2^{-m} \right) \|f\|_{h_{\rho,N}^p(\omega)} \leq C\|f\|_{h_{\rho,N}^p(\omega)},$$

moreover, we also have

$$\sum_{m \in \mathbb{N}} \sum_{i \in \mathbb{N}} |\lambda_i^m|^p + |\tilde{\lambda}_0|^p \leq C \left(\sum_{m \in \mathbb{N}} \|f_m\|_{h_{\rho,N}^p(\omega)}^p + \|f\|_{h_{\rho,N}^p(\omega)}^p \right) \leq C\|f\|_{h_{\rho,N}^p(\omega)}^p.$$

Finally, we obtain

$$f = \sum_{m \in \mathbb{N}} \sum_{i \in \mathbb{N}} \lambda_i^m a_i^m + \tilde{\lambda}_0 \tilde{a}_0 \in h_{\rho}^{p, \infty, s}(\omega)$$

and

$$\|f\|_{h_{\rho}^{p, \infty, s}(\omega)} \leq C \|f\|_{h_{\rho, N}^p(\omega)}.$$

The theorem is proved. \square

For simplicity, from now on, we denote by $h_{\rho}^p(\omega)$ the weighted local Hardy space $h_{\rho, N}^p(\omega)$ when $N \geq N_{p, \omega}$.

6 Finite atomic decompositions

In this section, we prove that for any given finite linear combination of weighted atoms when $q < \infty$, its norm in $h_{\rho, N}^p(\omega)$ can be achieved via all its finite weighted atomic decompositions. This extends the main results in [21] to the setting of weighted local Hardy spaces.

Definition 6.1. Let $\omega \in A_{\infty}^{\rho, \infty}(\mathbb{R}^n)$ and $(p, q, s)_{\omega}$ be admissible as in Definition 3.3. Then $h_{\rho, \text{fin}}^{p, q, s}(\omega)$ is defined to be the vector space of all finite linear combinations of $(p, q, s)_{\omega}$ -atoms and a $(p, q)_{\omega}$ -single-atom, and the norm of f in $h_{\rho, \text{fin}}^{p, q, s}(\omega)$ is defined by

$$\|f\|_{h_{\rho, \text{fin}}^{p, q, s}(\omega)} \equiv \inf \left\{ \left[\sum_{i=0}^k |\lambda_i|^p \right]^{1/p} : f = \sum_{i=0}^k \lambda_i a_i, \ k \in \mathbb{Z}_+, \ \{\lambda_i\}_{i=0}^k \subset \mathbb{C}, \ \{a_i\}_{i=1}^k \text{ are } (p, q, s)_{\omega} \text{ atoms, and } a_0 \text{ is a } (p, q)_{\omega} \text{ single atom} \right\}.$$

Obviously, for any admissible triplet $(p, q, s)_{\omega}$ atom and $(p, q)_{\omega}$ single atom, $h_{\rho, \text{fin}}^{p, q, s}(\omega)$ is dense in $h_{\rho}^{p, q, s}(\omega)$ with respect to the quasi-norm $\|\cdot\|_{h_{\rho}^{p, q, s}(\omega)}$.

Theorem 6.1. Let $\omega \in A_{\infty}^{\rho, \infty}(\mathbb{R}^n)$, q_{ω} be as in (2.4) and $(p, q, s)_{\omega}$ be admissible as in Definition 3.3. If $q \in (q_{\omega}, \infty)$, then $\|\cdot\|_{h_{\rho, \text{fin}}^{p, q, s}(\omega)}$ and $\|\cdot\|_{h_{\rho}^p(\omega)}$ are equivalent quasi-norms on $h_{\rho, \text{fin}}^{p, q, s}(\omega)$.

Proof. Obviously, by Theorem 5.1, we infer that $h_{\rho, \text{fin}}^{p, q, s}(\omega) \subset h_{\rho}^{p, q, s}(\omega) = h_{\rho}^p(\omega)$ and for all $f \in h_{\rho, \text{fin}}^{p, q, s}(\omega)$, we have

$$\|f\|_{h_{\rho}^p(\omega)} \leq C \|f\|_{h_{\rho, \text{fin}}^{p, q, s}(\omega)}.$$

Thus, it suffices to show that for every $q \in (q_{\omega}, \infty)$ there exists a constant C such that for all $f \in h_{\rho, \text{fin}}^{p, q, s}(\omega)$,

$$\|f\|_{h_{\rho, \text{fin}}^{p, q, s}(\omega)} \leq C \|f\|_{h_{\rho}^p(\omega)}. \quad (6.1)$$

Suppose that f is in $h_{\rho, \text{fin}}^{p, q, s}(\omega)$ with $\|f\|_{h_{\rho}^p(\omega)} = 1$. In this section, as in Section 5, we take $m_0 \in \mathbb{Z}$ such that $2^{m_0-1} \leq \inf_{x \in \mathbb{R}^n} \mathcal{M}_N f(x) < 2^{m_0}$, and for $\inf_{x \in \mathbb{R}^n} \mathcal{M}_N f(x) = 0$, write $m_0 = -\infty$. For each integer $m \geq m_0$, set

$$\Omega_m \equiv \{x \in \mathbb{R}^n : \mathcal{M}_N f(x) > 2^m\},$$

where and in what follows $N = N_{p, \omega}$. Since $f \in (h_{\rho, N}^p(\omega) \cap L_{\omega}^q(\mathbb{R}^n))$, by Lemma 5.4, there exist $\lambda_0 \in \mathbb{C}$, $\{\lambda_i^m\}_{m \geq k_0, i} \subset \mathbb{C}$, a $(p, \infty)_{\omega}$ -single-atom a_0 and $(p, \infty, s)_{\omega}$ -atoms $\{a_i^m\}_{m \geq m_0, i}$, such that

$$f = \sum_{m \geq m_0} \sum_i \lambda_i^m a_i^m + \lambda_0 a_0 \quad (6.2)$$

holds both in $\mathcal{D}'(\mathbb{R}^n)$ and almost everywhere. First, we claim that (6.2) also holds in $L_{\omega}^q(\mathbb{R}^n)$. For any $x \in \mathbb{R}^n$, by $\mathbb{R}^n = \cup_{m \geq m_0} (\Omega_{2^m} \setminus \Omega_{2^{m+1}})$, we see that there exists $j \in \mathbb{Z}$ such that $x \in (\Omega_{2^j} \setminus \Omega_{2^{j+1}})$. By the proof of Lemma 5.4, we know that for all $m > j$, $\text{supp}(a_i^m) \subset \tilde{Q}_i^m \subset \Omega_m \subset \Omega_{j+1}$; then from (5.13) and (5.18), we conclude that

$$\left| \sum_{m \geq m_0} \sum_i \lambda_i^m a_i^m(x) \right| + |\lambda_0 a_0(x)| \leq C \sum_{k_0 \leq k \leq j} 2^k + 2^{k_0} \leq C 2^j \leq C \mathcal{M}_N(f)(x).$$

Since $f \in L_{\omega}^q(\mathbb{R}^n)$, from Proposition 3.1(ii), we infer that $\mathcal{M}_N(f)(x) \in L_{\omega}^q(\mathbb{R}^n)$. This combined with the Lebesgue dominated convergence theorem implies that

$$\sum_{m \geq m_0} \sum_i \lambda_i^m a_i^m + \lambda_0 a_0$$

converges to f in $L_{\omega}^q(\mathbb{R}^n)$, which deduce the claim.

Next, we show (6.1) by considering the following two cases for ω .

Case I: $\omega(\mathbb{R}^n) = \infty$. In this case, as $f \in L_{\omega}^q(\mathbb{R}^n)$, we know that $m_0 = -\infty$ and $a_0(x) = 0$ for almost every $x \in \mathbb{R}^n$ in (6.2). Thus, in this case, (6.2) can be written as

$$f = \sum_{m \in \mathbb{Z}} \sum_i \lambda_i^m a_i^m.$$

Since, when $\omega(\mathbb{R}^n) = \infty$, all $(\rho, q)_{\omega}$ -single-atoms are 0, which implies that f has compact support for $f \in h_{\rho, \text{fin}}^{p, q, s}(\omega)$. Assume that $\text{supp}(f) \subset Q_0 \equiv Q(x_0, r_0)$ and $\tilde{Q}_0 \equiv Q(x_0, r_1)$, in which $r_1 = \sqrt{n}r_0 + C_0^2(1+R)^{k_0+1}(1 + \sqrt{n}r_0/\rho(x_0))\rho(x_0)$. Then for any $\psi \in \mathcal{D}_N(\mathbb{R}^n)$, $x \in \mathbb{R}^n \setminus \tilde{Q}_0$ and $2^{-l} \in (0, \rho(x))$, we have

$$\psi_l * f(x) = \int_{Q(x_0, r_0)} \psi_l(x-y) f(y) dy = \int_{B(x, R\rho(x)) \cap Q(x_0, r_0)} \psi_l(x-y) f(y) dy = 0.$$

Thus, for any $m \in \mathbb{Z}$, $\Omega_m \subset \tilde{Q}_0$, which implies that $\text{supp}(\sum_{m \in \mathbb{Z}} \sum_i \lambda_i^m a_i^m) \subset \tilde{Q}_0$. For each positive integer K , let

$$F_K \equiv \{(m, i) : m \in \mathbb{Z}, m \geq m_0, i \in \mathbb{N}, |m| + i \leq K\},$$

and

$$f_K \equiv \sum_{(m,i) \in F_K} \lambda_i^m a_i^m.$$

Then, by the above claim, we know that f_K converges to f in $L_\omega^q(\mathbb{R}^n)$. Hence, for any given $\varepsilon \in (0, 1)$, there exists a $K_0 \in \mathbb{N}$ large enough such that $\text{supp}(f - f_{K_0})/\varepsilon \subset \tilde{Q}_0$ and

$$\|(f - f_{K_0})/\varepsilon\|_{L_\omega^q(\mathbb{R}^n)} \leq [\omega(\tilde{Q}_0)]^{1/q-1/p}.$$

For \tilde{Q}_0 , since $l(\tilde{Q}_0) = r_1 > 2\rho(x_0)$, we can decompose it into finite disjoint cubes $\{Q_j\}_j$ such that $\tilde{Q}_0 = \bigcup_{j=1}^{N_0} Q_j$ and $l_j/4 < \rho(x) \leq C_0(3\sqrt{n})^{k_0} l_j$ for some $x \in Q_j = Q(x_j, l_j)$. Moreover, each l_j satisfies $L_2\rho(x_j) < l_j < L_1\rho(x_j)$. It is clear that for $q \in (q_\omega, \infty)$ and $p \in (0, 1]$ we have

$$\|(f - f_{K_0})\chi_{Q_j}/\varepsilon\|_{L_\omega^q(\mathbb{R}^n)} \leq [\omega(\tilde{Q}_0)]^{1/q-1/p} \leq [\omega(Q_j)]^{1/q-1/p},$$

which together with $\text{supp}((f - f_{K_0})\chi_{Q_j}/\varepsilon) \subset Q_j$ implies that $(f - f_{K_0})\chi_{Q_j}/\varepsilon$ is a $(p, q, s)_\omega$ -atom for $j = 1, 2, \dots, N_0$. Therefore,

$$f = f_{K_0} + \sum_{j=1}^{N_0} \varepsilon \frac{(f - f_{K_0})\chi_{Q_j}}{\varepsilon}$$

is a finite weighted atom linear combination of f almost everywhere. Then take $\varepsilon \equiv N_0^{-1/p}$, we obtain

$$\|f\|_{h_{\rho, \text{fin}}^{p, q, s}(\omega)}^p \leq \sum_{(m,i) \in F_K} |\lambda_i^m|^p + N_0 \varepsilon^p \leq C,$$

which implies the Case I.

Case II: $\omega(\mathbb{R}^n) < \infty$. In this case, f may not have compact support. Similarly to Case I, for any positive integer K , let

$$f_K \equiv \sum_{(m,i) \in F_K} \lambda_i^m a_i^m + \lambda_0 a_0$$

and $b_K \equiv f - f_K$, where F_K is as in Case I. From the above claim, we deduce that f_K converges to f in $L_\omega^q(\mathbb{R}^n)$. Thus, there exists a positive integer $K_1 \in \mathbb{N}$ large enough such that

$$\|b_{K_1}\|_{L_\omega^q(\mathbb{R}^n)} \leq [\omega(\mathbb{R}^n)]^{1/q-1/p}.$$

Thus, b_{K_1} is a $(p, q)_\omega$ -single-atom and $f = f_{K_1} + b_{K_1}$ is a finite weighted atom linear combination of f . By Lemma 5.4, we have

$$\|f\|_{h_{\rho, \text{fin}}^{p, q, s}(\omega)}^p \leq C \left(\sum_{(m,i) \in F_K} |\lambda_i^m|^p + \lambda_0^p \right) \leq C.$$

Thus, (6.1) holds, and the theorem is now proved. □

As an application of finite atomic decompositions, we establish boundedness in $h_\rho^p(\omega)$ of quasi-Banach-valued sublinear operators.

As in [5], a quasi-Banach space \mathcal{B} is a vector space endowed with a quasi-norm $\|\cdot\|_{\mathcal{B}}$ which is nonnegative, non-degenerate (i.e., $\|f\|_{\mathcal{B}} = 0$ if and only if $f = 0$), homogeneous, and obeys the quasi-triangle inequality, i.e., there exists a positive constant K no less than 1 such that for all $f, g \in \mathcal{B}$, $\|f + g\|_{\mathcal{B}} \leq K(\|f\|_{\mathcal{B}} + \|g\|_{\mathcal{B}})$.

Let $\beta \in (0, 1]$. A quasi-Banach space \mathcal{B}_β with the quasi-norm $\|\cdot\|_{\mathcal{B}_\beta}$ is called a β -quasi-Banach space if $\|f + g\|_{\mathcal{B}_\beta}^\beta \leq \|f\|_{\mathcal{B}_\beta}^\beta + \|g\|_{\mathcal{B}_\beta}^\beta$ for all $f, g \in \mathcal{B}_\beta$.

Notice that any Banach space is a 1-quasi-Banach space, and the quasi-Banach space l^β , $L_\omega^\beta(\mathbb{R}^n)$ and $h_\omega^\beta(\mathbb{R}^n)$ with $\beta \in (0, 1)$ are typical β -quasi-Banach spaces.

For any given β -quasi-Banach space \mathcal{B}_β with $\beta \in (0, 1]$ and a linear space \mathcal{Y} , an operator T from \mathcal{Y} to \mathcal{B}_β is said to be \mathcal{B}_β -sublinear if for any $f, g \in \mathcal{B}_\beta$ and $\lambda, \nu \in \mathbb{C}$,

$$\|T(\lambda f + \nu g)\|_{\mathcal{B}_\beta} \leq \left(|\lambda|^\beta \|T(f)\|_{\mathcal{B}_\beta}^\beta + |\nu|^\beta \|T(g)\|_{\mathcal{B}_\beta}^\beta \right)^{1/\beta}$$

and $\|T(f) - T(g)\|_{\mathcal{B}_\beta} \leq \|T(f - g)\|_{\mathcal{B}_\beta}$.

We remark that if T is linear, then it is \mathcal{B}_β -sublinear. Moreover, if \mathcal{B}_β is a space of functions, and T is nonnegative and sublinear in the classical sense, then T is also \mathcal{B}_β -sublinear.

Theorem 6.2. *Let $\omega \in A_\infty^{p,\infty}(\mathbb{R}^n)$, $0 < p \leq \beta \leq 1$, and \mathcal{B}_β be a β -quasi-Banach space. Suppose $q \in (q_\omega, \infty)$ and $T : h_{\rho, \text{fin}}^{p,q,s}(\omega) \rightarrow \mathcal{B}_\beta$ is a \mathcal{B}_β -sublinear operator such that*

$$S \equiv \sup\{\|T(a)\|_{\mathcal{B}_\beta} : a \text{ is a } (p, q, s)_\omega \text{ atom or } (p, q)_\omega \text{ single atom}\} < \infty.$$

Then there exists a unique bounded \mathcal{B}_β -sublinear operator \tilde{T} from $h_\rho^p(\omega)$ to \mathcal{B}_β which extends T .

Proof. For any $f \in h_{\rho, \text{fin}}^{p,q,s}(\omega)$, by Theorem 6.1, there exist a set of numbers $\{\lambda_j\}_{j=0}^l \subset \mathbb{C}$, $(p, q, s)_\omega$ -atoms $\{a_j\}_{j=1}^l$ and a $(p, q)_\omega$ single atom a_0 such that $f = \sum_{j=0}^l \lambda_j a_j$ pointwise and

$$\sum_{j=0}^l |\lambda_j|^p \leq C \|f\|_{h_\rho^p(\omega)}^p.$$

Then by the assumption, we have

$$\|T(f)\|_{\mathcal{B}_\beta} \leq C \left[\sum_{j=0}^l |\lambda_j|^p \right]^{1/p} \leq C \|f\|_{h_\rho^p(\omega)}.$$

Since $h_{\rho, \text{fin}}^{p,q,s}(\omega)$ is dense in $h_\rho^p(\omega)$, a density argument gives the desired results. \square

7 Atomic characterization of $H_{\mathcal{L}}^1(\omega)$

In this section, we apply the atomic characterization of the weighted local Hardy spaces $h_{\rho}^1(\omega)$ with $A_1^{\rho, \theta}(\mathbb{R}^n)$ weights to establish atomic characterization of weighted Hardy space $H_{\mathcal{L}}^1(\omega)$ associated to Schrödinger operator with $A_1^{\rho, \theta}(\mathbb{R}^n)$ weights.

Let $\mathcal{L} = -\Delta + V$ be a Schrödinger operator on \mathbb{R}^n , $n \geq 3$, where $V \in RH_{n/2}$ is a fixed non-negative potential.

Let $\{T_t\}_{t>0}$ be the semigroup of linear operators generated by \mathcal{L} and $T_t(x, y)$ be their kernels, that is,

$$T_t f(x) = e^{-t\mathcal{L}} f(x) = \int_{\mathbb{R}^n} T_t(x, y) f(y) dy, \quad \text{for } t > 0 \text{ and } f \in L^2(\mathbb{R}^n). \quad (7.1)$$

Since V is non-negative the Feynman-Kac formula implies that

$$0 \leq T_t(x, y) \leq \tilde{T}_t(x, y) \equiv (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{|x-y|^2}{4t}\right). \quad (7.2)$$

Obviously, by (1.2) the maximal operator

$$\mathcal{T}^* f(x) = \sup_{t>0} |T_t f(x)|$$

is of weak-type (1,1). A weighted Hardy-type space related to \mathcal{L} with $A_1^{\rho, \theta}(\mathbb{R}^n)$ weights is naturally defined by:

$$H_{\mathcal{L}}^1(\omega) \equiv \{f \in L_{\omega}^1(\mathbb{R}^n) : \mathcal{T}^* f(x) \in L_{\omega}^1(\mathbb{R}^n)\}, \quad \text{with } \|f\|_{H_{\mathcal{L}}^1(\omega)} \equiv \|\mathcal{T}^* f\|_{L_{\omega}^1(\mathbb{R}^n)}. \quad (7.3)$$

The $H_{\mathcal{L}}^1(\omega)$ with $\omega \in A_1(\mathbb{R}^n)$ has been studied in [19, 41]

Now let us recall some basic properties of kernels $T_t(x, y)$ and the operator \mathcal{T}^*

Lemma 7.1. (see [10]) *For every $l > 0$ there is a constant C_l such that*

$$T_t(x, y) \leq C_l (1 + |x-y|/\rho(x))^{-l} |x-y|^{-n}, \quad (7.4)$$

for $x, y \in \mathbb{R}^n$. Moreover, there is an $\varepsilon > 0$ such that for every $C' > 0$, there exists C so that

$$|T_t(x, y) - \tilde{T}_t(x, y)| \leq C \frac{(|x-y|/\rho(x))^{\varepsilon}}{|x-y|^n}, \quad (7.5)$$

for $|x-y| \leq C' \rho(x)$.

Since $T_t(x, y)$ is a symmetric function, we also have

$$T_t(x, y) \leq C_l (1 + |x-y|/\rho(y))^{-l} |x-y|^{-n}, \quad \text{for } x, y \in \mathbb{R}^n. \quad (7.6)$$

Lemma 7.2. (see [11]) *There exist a rapidly decaying function $w \geq 0$ and a $\delta > 0$ such that*

$$|T_t(x, y) - \tilde{T}_t(x, y)| \leq \left(\frac{\sqrt{t}}{\rho(x)} \right)^\delta w_{\sqrt{t}}(x - y). \quad (7.7)$$

Lemma 7.3. (see [12]) *If $V \in RH_s(\mathbb{R}^n)$, $s > n/2$, then there exist $\delta = \delta(s) > 0$ and $c > 0$ such that for every $N > 0$, there is a constant C_N so that, for all $|h| < \sqrt{t}$*

$$|T_t(x + h, y) - T_t(x, y)| \leq C_N \left(\frac{|h|}{\sqrt{t}} \right)^\delta t^{-\frac{n}{2}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-N} \exp \left(-\frac{c|x - y|^2}{t} \right). \quad (7.8)$$

Lemma 7.4. (see [3]) *For $1 < p < \infty$ the operator \mathcal{T}^* is bounded on $L^p(\omega)$ when $\omega \in A_p^{\rho, \infty}(\mathbb{R}^n)$, and of weak type $(1, 1)$ when $\omega \in A_1^{\rho, \infty}(\mathbb{R}^n)$.*

In order to achieve the desired conclusions, we need the following estimates.

Lemma 7.5. *Let $\omega \in A_1^{\rho, \infty}(\mathbb{R}^n)$, then there exists a positive constant C such that for all $f \in h_\rho^1(\omega)$,*

$$\|f\|_{h_\rho^1(\omega)} \leq C \|\tilde{T}_\rho^+(f)\|_{L_\omega^1(\mathbb{R}^n)}, \quad (7.9)$$

where

$$\tilde{T}_\rho^+(f)(x) \equiv \sup_{0 < t < \rho(x)} |\tilde{T}_{t^2}(f)(x)|$$

and

$$\tilde{T}_t(f)(x) \equiv \int_{\mathbb{R}^n} \tilde{T}_t(x, y) f(y) dy.$$

Proof. Let $h(x) = (4\pi)^{-n/2} e^{-|x|^2/4}$, then it is easy to find that $h_t(x - y) = \tilde{T}_{t^2}(x, y)$. Now we take a nonnegative function $\varphi \in \mathcal{D}(\mathbb{R}^n)$ such that $\varphi(x) = h(x)$ on $B(0, 2)$, and we define $\varphi_\rho^+(f)(x)$ as follows:

$$\varphi_\rho^+(f)(x) \equiv \sup_{0 < t < \rho(x)} |\varphi_t * f(x)|.$$

Clearly, for any $x \in \mathbb{R}^n$, we have

$$\varphi^+(f)(x) \leq \varphi_\rho^+(f)(x), \quad (7.10)$$

see (3.4) for the definition of $\varphi^+(f)(x)$.

Let $f \in h_\rho^1(\omega)$, for every $N > 0$ we have:

$$\begin{aligned} & \left\| \varphi_\rho^+(f) - \tilde{T}_\rho^+(f) \right\|_{L_\omega^1(\mathbb{R}^n)} \\ & \leq \int_{\mathbb{R}^n} \sup_{0 < t < \rho(x)} |\varphi_t * f(x) - h_t * f(x)| \omega(x) dx \end{aligned}$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}^n} \left(\sup_{0 < t < \rho(x)} t^{-n} \int_{\mathbb{R}^n} |f(y)| \left| \varphi \left(\frac{x-y}{t} \right) - h \left(\frac{x-y}{t} \right) \right| dy \right) \omega(x) dx \\
&\leq \int_{\mathbb{R}^n} \left(\sup_{0 < t < \rho(x)} t^{-n} \int_{\mathbb{R}^n} |f(y)| \left| \varphi \left(\frac{x-y}{t} \right) - h \left(\frac{x-y}{t} \right) \right| \chi_{\{|y-x|>t\}}(y) dy \right) \omega(x) dx \\
&\leq C \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(y)| \sup_{0 < t < \rho(x)} t^{-n} \left(1 + \frac{|x-y|}{t} \right)^{-N} \chi_{\{|y-x|>t\}}(y) dy \right) \omega(x) dx \\
&\leq C \int_{\mathbb{R}^n} |f(y)| \left(\int_{\mathbb{R}^n} (\rho(x))^{-n} \left(1 + \frac{|x-y|}{\rho(x)} \right)^{-N} \omega(x) dx \right) dy.
\end{aligned}$$

In the last inequality, we used the following facts that

$$\sup_{0 < t < \rho(x)} t^{-n} \left(1 + \frac{|x-y|}{t} \right)^{-N} \leq (\rho(x))^{-n} \left(1 + \frac{|x-y|}{\rho(x)} \right)^{-N},$$

provided that $|x-y| > t$ and $N > 2n$.

We now estimate the inner integral in the last inequality. In fact,

$$\begin{aligned}
&\int_{\mathbb{R}^n} (\rho(x))^{-n} \left(1 + \frac{|x-y|}{\rho(x)} \right)^{-N} \omega(x) dx \\
&= \int_{|x-y| < \rho(y)} (\rho(x))^{-n} \left(1 + \frac{|x-y|}{\rho(x)} \right)^{-N} \omega(x) dx \\
&\quad + \int_{|x-y| \geq \rho(y)} (\rho(x))^{-n} \left(1 + \frac{|x-y|}{\rho(x)} \right)^{-N} \omega(x) dx \\
&\equiv I + II.
\end{aligned}$$

For I , since N is large enough and (2.2), we have

$$I \leq \frac{C}{(\rho(y))^n} \int_{|x-y| < \rho(y)} \omega(x) dx \leq C \Psi_\theta(\tilde{B}_0) M_{V,\theta}(\omega)(y) \leq C \omega(y),$$

where $\tilde{B}_0 = B(y, \rho(y))$.

For II , by the same reason as above, we have

$$\begin{aligned}
II &\leq C \sum_{i=1}^{\infty} \int_{|x-y| \sim 2^i \rho(y)} (\rho(x))^{N-n} |x-y|^{-N} \omega(x) dx \\
&\leq C \sum_{i=1}^{\infty} \int_{|x-y| \sim 2^i \rho(y)} (\rho(y))^{N-n} \left(1 + \frac{|x-y|}{\rho(y)} \right)^{\frac{k_0(N-n)}{k_0+1}} |x-y|^{-N} \omega(x) dx \\
&\leq C \sum_{i=1}^{\infty} \int_{|x-y| \sim 2^i \rho(y)} (\rho(y))^{N-n} (1+2^i)^{\frac{k_0(N-n)}{k_0+1}} (2^i \rho(y))^{-N} \omega(x) dx
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{i=1}^{\infty} (2^{-i})^{\frac{N+nk_0}{k_0+1}} \frac{1}{(\rho(y))^n} \int_{|x-y| < 2^i \rho(y)} \omega(x) dx \\
&\leq C \sum_{i=1}^{\infty} (2^{-i})^{\frac{N+nk_0}{k_0+1}} (1+2^i)^{\theta} M_{V,\theta}(\omega)(y) \\
&\leq C \sum_{i=1}^{\infty} (2^{-i})^{\frac{N+nk_0}{k_0+1}-\theta} \omega(y) \leq C\omega(y),
\end{aligned}$$

and the last inequality holds because the real number N is large enough.

Combining the above two estimates, we get

$$\left\| \varphi_{\rho}^{+}(f) - \tilde{T}_{\rho}^{+}(f) \right\|_{L_{\omega}^1(\mathbb{R}^n)} \leq C \int_{\mathbb{R}^n} |f(y)| \omega(y) dy = C \|f\|_{L_{\omega}^1(\mathbb{R}^n)}. \quad (7.11)$$

In addition, it is easy to get $\|f\|_{L_{\omega}^1(\mathbb{R}^n)} \leq \|\tilde{T}_{\rho}^{+} f\|_{L_{\omega}^1(\mathbb{R}^n)}$. Therefore, we obtain

$$\left\| \varphi_{\rho}^{+}(f) \right\|_{L_{\omega}^1(\mathbb{R}^n)} \leq \left\| \tilde{T}_{\rho}^{+}(f) \right\|_{L_{\omega}^1(\mathbb{R}^n)} + C \|f\|_{L_{\omega}^1(\mathbb{R}^n)} \leq C \left\| \tilde{T}_{\rho}^{+}(f) \right\|_{L_{\omega}^1(\mathbb{R}^n)}. \quad (7.12)$$

Finally, from Theorem 3.2, (7.10) and (7.12), it follows that

$$\|f\|_{h_{\rho}^1(\omega)} \leq C \left\| \varphi_{\rho}^{+}(f) \right\|_{L_{\omega}^1(\mathbb{R}^n)} \leq C \left\| \varphi_{\rho}^{+}(f) \right\|_{L_{\omega}^1(\mathbb{R}^n)} \leq C \left\| \tilde{T}_{\rho}^{+}(f) \right\|_{L_{\omega}^1(\mathbb{R}^n)},$$

which finishes the proof. \square

For $x, y \in \mathbb{R}^n$, set $E_t(x, y) = T_{t^2}(x, y) - \tilde{T}_{t^2}(x, y)$,

$$T_{\rho}^{+}(f)(x) \equiv \sup_{0 < t < \rho(x)} |T_{t^2}(f)(x)| \quad \text{and} \quad E_{\rho}^{+}(f)(x) \equiv \sup_{0 < t < \rho(x)} |E_t(f)(x)|.$$

Lemma 7.6. *Let $\omega \in A_1^{\rho, \infty}(\mathbb{R}^n)$, then there exists a positive constant C such that for all $f \in L_{\omega}^1(\mathbb{R}^n)$,*

$$\|E_{\rho}^{+}(f)\|_{L_{\omega}^1(\mathbb{R}^n)} \leq C \|f\|_{L_{\omega}^1(\mathbb{R}^n)}.$$

Proof. By Lemma 2.2, it suffices to prove that for all j ,

$$\|E_{\rho}^{+}(\chi_{B_j^*} f)\|_{L_{\omega}^1(\mathbb{R}^n)} \leq C \|\chi_{B_j^*} f\|_{L_{\omega}^1(\mathbb{R}^n)}, \quad (7.13)$$

in which $B_j = B(x_j, \rho(x_j))$. For any $x \in B_j^{**}$ and $y \in B_j^*$, since $\rho(y) \sim \rho(x_j) \sim \rho(x)$ via Lemma 2.1, by (7.5) we have

$$|E_t(x, y)| \leq C \frac{(|x-y|/\rho(x))^{\varepsilon}}{|x-y|^n} \leq \frac{C}{|x-y|^{n-\varepsilon}(\rho(x_j))^{\varepsilon}},$$

which implies that

$$\begin{aligned}
& \int_{B_j^{**}} \sup_{0 < t < \rho(x)} |E_t(\chi_{B_j^*} f)| \omega(x) dx \\
& \leq C \int_{B_j^{**}} \left(\int_{B_j^*} \frac{|f(y)|}{|x-y|^{n-\varepsilon}(\rho(x_j))^\varepsilon} dy \right) \omega(x) dx \\
& \leq C \int_{B_j^*} \left(\int_{B_j^{**}} \frac{\omega(x)}{|x-y|^{n-\varepsilon}(\rho(x_j))^\varepsilon} dx \right) |f(y)| dy \\
& \leq C \int_{B_j^*} \left(\sum_{k=-2}^{\infty} \int_{|x-y| \sim 2^{-k}\rho(x_j)} \frac{\omega(x)}{|x-y|^{n-\varepsilon}(\rho(x_j))^\varepsilon} dx \right) |f(y)| dy \\
& \leq C \int_{B_j^*} \left(\sum_{k=-2}^{\infty} \frac{\omega(B(y, 2^{-k}\rho(x_j)))}{(2^{-k}\rho(x_j))^{n-\varepsilon}(\rho(x_j))^\varepsilon} dx \right) |f(y)| dy \\
& \leq C \int_{B_j^*} \left(\sum_{k=-2}^{\infty} \frac{1}{2^{k\varepsilon}} \left(1 + C_0 2^{k_0-k}\right)^\theta \omega(y) \right) |f(y)| dy \\
& \leq C \int_{B_j^*} |f(y)| \omega(y) dy = C \|\chi_{B_j^*} f\|_{L_\omega^1(\mathbb{R}^n)}.
\end{aligned}$$

For any $x \in (B_j^{**})^\complement$ and $y \in B_j^*$, it is easy to see that $\rho(x_j) \lesssim |x - x_j| \sim |x - y|$; in addition, by (2.2) and (7.7), we have $0 < t < \rho(x) \lesssim |x - x_j|^{k_0/(k_0+1)}(\rho(x_j))^{1/(k_0+1)}$ and $E_t(x, y) \lesssim t^N/|x - y|^{N+n} \sim t^N/|x - x_j|^{N+n}$ for any $N > 0$. Therefore, taking $N > (k_0 + 1)\theta$, we have

$$\begin{aligned}
& \int_{(B_j^{**})^\complement} \sup_{0 < t < \rho(x)} |E_t(\chi_{B_j^*} f)| \omega(x) dx \\
& \leq C \int_{(B_j^{**})^\complement} \left(\int_{B_j^*} \frac{(\rho(x_j))^{\frac{N}{k_0+1}} |f(y)|}{|x - x_j|^{n+\frac{N}{k_0+1}}} dy \right) \omega(x) dx \\
& \leq C \int_{B_j^*} \left(\int_{(B_j^{**})^\complement} \frac{(\rho(x_j))^{\frac{N}{k_0+1}} \omega(x)}{|x - x_j|^{n+\frac{N}{k_0+1}}} dx \right) |f(y)| dy \\
& \leq C \int_{B_j^*} \left(\sum_{i=2}^{\infty} \int_{|x-x_j| \sim 2^i \rho(x_j)} \frac{(\rho(x_j))^{\frac{N}{k_0+1}} \omega(x)}{|x - x_j|^{n+\frac{N}{k_0+1}}} dx \right) |f(y)| dy \\
& \leq C \int_{B_j^*} \left(\sum_{i=2}^{\infty} \frac{(\rho(x_j))^{\frac{N}{k_0+1}} \omega(B(x_j, 2^i \rho(x_j)))}{(2^i \rho(x_j))^{n+\frac{N}{k_0+1}}} dx \right) |f(y)| dy \\
& \leq C \int_{B_j^*} \left(\sum_{i=2}^{\infty} \frac{(1 + 2^i)^\theta}{(2^i)^{\frac{N}{k_0+1}}} \omega(y) \right) |f(y)| dy \\
& \leq C \int_{B_j^*} |f(y)| \omega(y) dy = C \|\chi_{B_j^*} f\|_{L_\omega^1(\mathbb{R}^n)},
\end{aligned}$$

which completes the proof of (7.13) and hence the proof of lemma. \square

Next we give several estimates about $(p, q, s)_\omega$ -atoms and $(p, q)_\omega$ -single-atom, which are important for our conclusion.

Lemma 7.7. *Let a be a $(p, q, s)_\omega$ -atom, and $\text{supp } a \subset Q(x_0, r)$, then for any $x \in (4Q)^\complement$, we have following estimates:*

(i) *If $L_2\rho(x_0) \leq r \leq L_1\rho(x_0)$, then for any $M > 0$,*

$$\mathcal{T}^*a(x) \lesssim \|a\|_{L^1(\mathbb{R}^n)} \frac{r^M}{|x - x_0|^{n+M}},$$

(ii) *If $r < L_2\rho(x_0)$ and $|x - x_0| \leq 2\rho(x_0)$, then there exists $\delta > 0$ such that*

$$\mathcal{T}^*a(x) \lesssim \|a\|_{L^1(\mathbb{R}^n)} \frac{r^\delta}{|x - x_0|^{n+\delta}},$$

(iii) *If $r < L_2\rho(x_0)$ and $|x - x_0| \geq \rho(x_0)/\sqrt{n}$, then there exists $\delta > 0$ such that for any $M > 0$,*

$$\mathcal{T}^*a(x) \lesssim \|a\|_{L^1(\mathbb{R}^n)} \frac{r^\delta}{|x - x_0|^{n+\delta}} \left(\frac{\rho(x_0)}{|x - x_0|} \right)^M.$$

Proof. If $L_2\rho(x_0) \leq r \leq L_1\rho(x_0)$, since $|x - y| \sim |x - x_0|$ and $\rho(y) \sim \rho(x_0)$ for $x \in (4Q)^\complement$ and $y \in Q$, by Lemma 7.1, for any $M > 0$, we have

$$\begin{aligned} T_t a(x) &\leq \int_{\mathbb{R}^n} |T_t(x, y)| |a(y)| dy \\ &\lesssim \int_Q \left(1 + \frac{|x - y|}{\rho(y)} \right)^{-M} |x - y|^{-n} |a(y)| dy \\ &\lesssim \int_Q \left(1 + \frac{|x - x_0|}{\rho(x_0)} \right)^{-M} |x - x_0|^{-n} |a(y)| dy \\ &\lesssim \|a\|_{L^1(\mathbb{R}^n)} \frac{\rho(x_0)^M}{|x - x_0|^{n+M}} \lesssim \|a\|_{L^1(\mathbb{R}^n)} \frac{r^M}{|x - x_0|^{n+M}}, \end{aligned}$$

and then we obtain (i).

If $r < L_2\rho(x_0)$, by the moment condition of a and Lemma 7.3, for any $M > 0$ and $y' \in Q$ which satisfies $|y - y'| < \sqrt{t}$, we have

$$\begin{aligned} T_t a(x) &= \int_{\mathbb{R}^n} T_t(x, y) a(y) dy \\ &= \int_Q (T_t(x, y) - T_t(x, y')) a(y) dy \\ &\lesssim \int_Q \left(\frac{|y - y'|}{\sqrt{t}} \right)^\delta t^{-\frac{n}{2}} \left(1 + \frac{\sqrt{t}}{\rho(y)} \right)^{-M} \exp \left(-\frac{c|x - y|^2}{t} \right) |a(y)| dy \\ &\lesssim \int_Q \left(\frac{r}{\sqrt{t}} \right)^\delta t^{-\frac{n}{2}} \left(1 + \frac{\sqrt{t}}{\rho(x_0)} \right)^{-M} \left(\frac{t}{|x - x_0|^2} \right)^K |a(y)| dy, \end{aligned}$$

where $K > 0$ is any real number.

For $|x - x_0| \leq 2\rho(x_0)$, taking $K = (n + \delta)/2$, we obtain

$$\begin{aligned} T_t a(x) &\lesssim \int_Q \left(\frac{r}{\sqrt{t}} \right)^\delta t^{-\frac{n}{2}} \left(1 + \frac{\sqrt{t}}{\rho(x_0)} \right)^{-M} \left(\frac{t}{|x - x_0|^2} \right)^K |a(y)| dy, \\ &\lesssim \|a\|_{L^1(\mathbb{R}^n)} \left(\frac{r}{\sqrt{t}} \right)^\delta t^{-\frac{n}{2}} \left(\frac{t}{|x - x_0|^2} \right)^K \\ &= \|a\|_{L^1(\mathbb{R}^n)} \frac{r^\delta}{|x - x_0|^{n+\delta}}, \end{aligned}$$

which implies (ii).

For $|x - x_0| \geq \rho(x_0)/\sqrt{n}$, taking $K = (n + M + \delta)/2$, we obtain

$$\begin{aligned} T_t a(x) &\lesssim \int_Q \left(\frac{r}{\sqrt{t}} \right)^\delta t^{-\frac{n}{2}} \left(1 + \frac{\sqrt{t}}{\rho(x_0)} \right)^{-M} \left(\frac{t}{|x - x_0|^2} \right)^K |a(y)| dy, \\ &\lesssim \|a\|_{L^1(\mathbb{R}^n)} \left(\frac{r}{\sqrt{t}} \right)^\delta t^{-\frac{n}{2}} \left(\frac{\rho(x_0)}{\sqrt{t}} \right)^M \left(\frac{t}{|x - x_0|^2} \right)^K \\ &= \|a\|_{L^1(\mathbb{R}^n)} \frac{r^\delta}{|x - x_0|^{n+\delta}} \left(\frac{\rho(x_0)}{|x - x_0|} \right)^M, \end{aligned}$$

which finishes the proof of lemma. \square

Lemma 7.8. *Let $\omega \in A_q^{\rho, \theta}(\mathbb{R}^n)$ and a be a $(p, q, s)_\omega$ -atom, which satisfies $\text{supp } a \subset Q(x_0, r)$, then there exists a constant C such that:*

$$\|a\|_{L^1(\mathbb{R}^n)} \leq C|Q|\omega(Q)^{-1/p}\Psi_\theta(Q).$$

Proof. If $q > 1$, by Hölder inequality and the definition of $A_q^{\rho, \theta}(\mathbb{R}^n)$ weights, we have

$$\begin{aligned} \|a\|_{L^1(\mathbb{R}^n)} &= \int_Q |a(x)| \omega(x)^{1/q} \omega(x)^{-1/q} dx \\ &\leq \|a\|_{L_\omega^q(\mathbb{R}^n)} \left(\int_Q \omega(x)^{-q'/q} dx \right)^{1/q'} \\ &\leq \omega(Q)^{1/q-1/p} \left(\int_Q \omega(x)^{-q'/q} dx \right)^{1/q'} \left(\int_Q \omega(x) dx \right)^{1/q} \omega(Q)^{-1/q} \\ &\leq C|Q|\omega(Q)^{-1/p}\Psi_\theta(Q). \end{aligned}$$

If $q = 1$, we have

$$\omega(Q) \leq C|Q|\Psi_\theta(Q) \inf_{x \in Q} \omega(x),$$

which implies

$$\|\omega^{-1}\|_{L^\infty(Q)} \leq C|Q|\omega(Q)^{-1}\Psi_\theta(Q).$$

Therefore, we get

$$\|a\|_{L^1(\mathbb{R}^n)} \leq \|a\|_{L_\omega^1(\mathbb{R}^n)} \|\omega^{-1}\|_{L^\infty(Q)} \leq C|Q|\omega(Q)^{-1/p}\Psi_\theta(Q),$$

which finishes the proof. \square

Combining above two lemmas with $\Psi_\theta(Q) \lesssim 1$, we can get the following corollary.

Corollary 7.1. *Let a be a $(p, q, s)_\omega$ -atom, and $\text{supp } a \subset Q(x_0, r)$, then for any $x \in (4Q)^\complement$, we have following estimates:*

(i) *If $L_2\rho(x_0) \leq r \leq L_1\rho(x_0)$, then for any $M > 0$,*

$$\mathcal{T}^*a(x) \lesssim \omega(Q)^{-1/p} \left(\frac{r}{|x - x_0|} \right)^{n+M},$$

(ii) *If $r < L_2\rho(x_0)$ and $|x - x_0| \leq 2\rho(x_0)$, then there exists $\delta > 0$ such that*

$$\mathcal{T}^*a(x) \lesssim \omega(Q)^{-1/p} \left(\frac{r}{|x - x_0|} \right)^{n+\delta},$$

(iii) *If $r < L_2\rho(x_0)$ and $|x - x_0| \geq \rho(x_0)/\sqrt{n}$, then there exists $\delta > 0$ such that for any $M > 0$,*

$$\mathcal{T}^*a(x) \lesssim \omega(Q)^{-1/p} \left(\frac{r}{|x - x_0|} \right)^{n+\delta} \left(\frac{\rho(x_0)}{|x - x_0|} \right)^M.$$

Next we give the main theorem of this section.

Theorem 7.1. *Let $0 \neq V \in RH_{n/2}$ and $\omega \in A_1^{\rho, \infty}(\mathbb{R}^n)$, then $h_\rho^1(\omega) = H_\mathcal{L}^1(\omega)$ with equivalent norms, that is*

$$\|f\|_{h_\rho^1(\omega)} \sim \|f\|_{H_\mathcal{L}^1(\omega)}.$$

Proof. Assume that $f \in H_\mathcal{L}^1(\omega)$, by (7.7), we have

$$|f(x)| = \lim_{t < \rho(x), t \rightarrow 0} |\tilde{T}_t(f)(x)| \leq T_\rho^+(f)(x) + C \lim_{t \rightarrow 0} \left(\frac{t}{\rho(x)} \right)^\delta M(f)(x) \leq T_\rho^+(f)(x). \quad (7.14)$$

Then according to (7.14), Lemma 7.5 and Lemma 7.6, we get $f \in h_\rho^1(\omega)$ and

$$\begin{aligned} \|f\|_{h_\rho^1(\omega)} &\lesssim \|\tilde{T}_\rho^+(f)\|_{L_\omega^1(\mathbb{R}^n)} \lesssim \|T_\rho^+(f)\|_{L_\omega^1(\mathbb{R}^n)} + \|E_\rho^+(f)\|_{L_\omega^1(\mathbb{R}^n)} \\ &\lesssim \|T_\rho^+(f)\|_{L_\omega^1(\mathbb{R}^n)} + \|f\|_{L_\omega^1(\mathbb{R}^n)} \lesssim \|T_\rho^+(f)\|_{L_\omega^1(\mathbb{R}^n)} \\ &\lesssim \|\mathcal{T}^*(f)\|_{L_\omega^1(\mathbb{R}^n)} = \|f\|_{H_\mathcal{L}^1(\omega)}. \end{aligned}$$

Conversely, we need to prove that \mathcal{T}^* is bounded from $h_\rho^1(\omega)$ to $L_\omega^1(\mathbb{R}^n)$. To end this, by Lemma 2.4 and Theorem 5.1, it suffices to prove that for any $(1, q, s)_\omega$ -atom or $(1, q)_\omega$ -single-atom a ,

$$\|\mathcal{T}^*(a)\|_{L_\omega^1(\mathbb{R}^n)} \lesssim 1, \quad (7.15)$$

where $1 < q \leq 1 + \delta/n$.

If a is a $(1, q)_\omega$ -single-atom, by Hölder inequality and Lemma 7.4, we have

$$\|\mathcal{T}^*(a)\|_{L_\omega^1(\mathbb{R}^n)} \leq \|\mathcal{T}^*(a)\|_{L_\omega^q(\mathbb{R}^n)} \omega(\mathbb{R}^n)^{1-1/q} \leq C \|a\|_{L_\omega^q(\mathbb{R}^n)} \omega(\mathbb{R}^n)^{1-1/q} \lesssim 1.$$

If a is a $(1, q, s)_\omega$ -atom and $\text{supp } a \subset Q(x_0, r)$ with $r \leq L_1 \rho(x_0)$, then we have

$$\|\mathcal{T}^*(a)\|_{L_\omega^1(\mathbb{R}^n)} \leq \|\mathcal{T}^*(a)\|_{L_\omega^1(4Q)} + \|\mathcal{T}^*(a)\|_{L_\omega^1((4Q)^c)} \equiv I + II.$$

For I , by Hölder inequality, Lemma 2.4 and Lemma 7.4, we get

$$\begin{aligned} \|\mathcal{T}^*(a)\|_{L_\omega^1(4Q)} &\leq \|\mathcal{T}^*(a)\|_{L_\omega^q(4Q)} \omega(4Q)^{1-1/q} \leq C \|a\|_{L_\omega^q(\mathbb{R}^n)} \omega(4Q)^{1-1/q} \\ &\leq C (\omega(4Q)/\omega(Q))^{1-1/q} \lesssim 1. \end{aligned}$$

For II , if $L_2 \rho(x_0) \leq r \leq L_1 \rho(x_0)$, by Lemma 2.4 and Corollary 7.1, taking $M > q(n + \theta) - n$, we have

$$\begin{aligned} \|\mathcal{T}^*(a)\|_{L_\omega^1((4Q)^c)} &= \sum_{j=3}^{\infty} \int_{2^j Q \setminus 2^{j-1} Q} \mathcal{T}^*(a)(x) \omega(x) dx \\ &\lesssim \frac{1}{\omega(Q)} \sum_{j=3}^{\infty} \int_{2^j Q \setminus 2^{j-1} Q} \left(\frac{r}{|x - x_0|} \right)^{n+M} \omega(x) dx \\ &\lesssim \frac{1}{\omega(Q)} \sum_{j=3}^{\infty} 2^{-j(n+M)} \omega(2^j Q) \\ &\lesssim \sum_{j=3}^{\infty} 2^{-j(n+M)} 2^{jnq} \left(1 + \frac{2^j r}{\rho(x_0)} \right)^{q\theta} \\ &\lesssim \sum_{j=3}^{\infty} 2^{-j[n+M-nq-q\theta]} \lesssim 1; \end{aligned}$$

if $r < L_2 \rho(x_0)$, then there exists $N_0 \in \mathbb{Z}$ such that $2^{N_0-1} \sqrt{n} r \leq \rho(x_0) < 2^{N_0} \sqrt{n} r$. Let us assume that $N_0 \geq 3$, otherwise, we just need to consider the I_2 in the following decomposition:

$$\|\mathcal{T}^*(a)\|_{L_\omega^1((4Q)^c)} = \left(\sum_{j=3}^{N_0} + \sum_{j=N_0+1}^{\infty} \right) \int_{2^j Q \setminus 2^{j-1} Q} \mathcal{T}^*(a)(x) \omega(x) dx \equiv I_1 + I_2,$$

for I_1 , since $|x - x_0| < 2^j \sqrt{n}r \leq 2^{N_0} \sqrt{n}r \leq 2\rho(x_0)$, $\Psi_\theta(2^j Q) \leq 3^\theta$ and $q < 1 + \delta/n$, by Lemma 2.4 and Corollary 7.1, we get

$$\begin{aligned}
I_1 &= \sum_{j=3}^{N_0} \int_{2^j Q \setminus 2^{j-1} Q} \mathcal{T}^*(a)(x) \omega(x) dx \\
&\lesssim \frac{1}{\omega(Q)} \sum_{j=3}^{N_0} \int_{2^j Q \setminus 2^{j-1} Q} \left(\frac{r}{|x - x_0|} \right)^{n+\delta} \omega(x) dx \\
&\lesssim \frac{1}{\omega(Q)} \sum_{j=3}^{N_0} 2^{-j(n+\delta)} \omega(2^j Q) \\
&\lesssim \sum_{j=3}^{N_0} 2^{-j[n+\delta-nq]} \lesssim 1,
\end{aligned}$$

for I_2 , since $|x - x_0| \geq 2^{j-1}r \geq 2^{N_0}r \geq \rho(x_0)/\sqrt{n}$, then $\Psi_\theta(2^j Q) \leq (2^{j+1}\sqrt{n}r/\rho(x_0))^\theta$, thus, taking $M = q\theta$, by $q < 1 + \delta/n$, Lemma 2.4 and Corollary 7.1, we obtain

$$\begin{aligned}
I_2 &= \sum_{j=N_0+1}^{\infty} \int_{2^j Q \setminus 2^{j-1} Q} \mathcal{T}^*(a)(x) \omega(x) dx \\
&\lesssim \frac{1}{\omega(Q)} \sum_{j=N_0+1}^{\infty} \int_{2^j Q \setminus 2^{j-1} Q} \left(\frac{r}{|x - x_0|} \right)^{n+\delta} \left(\frac{\rho(x_0)}{|x - x_0|} \right)^M \omega(x) dx \\
&\lesssim \frac{1}{\omega(Q)} \sum_{j=N_0+1}^{\infty} 2^{-j(n+\delta)} \omega(2^j Q) \left(\frac{\rho(x_0)}{2^j r} \right)^M \\
&\lesssim \sum_{j=N_0+1}^{\infty} 2^{-j[n+\delta-nq]} (\Psi_\theta(2^j Q))^q \left(\frac{\rho(x_0)}{2^j r} \right)^M \lesssim 1,
\end{aligned}$$

which finally implies the (7.15) and finishes the proof. \square

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